Generic Finiteness of Equilibrium Distributions for Bimatrix Outcome Game Forms*

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1 Introduction

It has long been established (Rosenmüller (1971), Wilson (1971), Harsanyi (1973)) that for normal form games with an arbitrary number of players if the payoffs can be independently perturbed, there is generically a finite number of equilibria. For the case of extensive form games Kreps and Wilson (1982) show that the equilibrium distributions on terminal nodes are generically finite.

The equivalent result for outcome games has turned out to be difficult to elucidate. On the one hand, Govindan and McLennan (2001) were the first to provide an example of a game form for which the Nash equilibria of the games associated to an open set of utility profiles induce a continuum of probability distributions on outcomes. Their example made use of three players and six outcomes. Examples of game forms with the same feature have appeared since: Kukushkin, Litan, and Marhuenda (2008) with two players and four outcomes and Litan, Marhuenda, and Sudhölter (2015) with three players and three outcomes.

On the other hand, there are a number of results that point towards the paucity of such examples. Govindan and McLennan (2001) proved that for games with two outcomes and any number of players the number of equilibrium distributions on outcomes is generically finite. Similar results have been obtained for two player, three outcomes games (González-Pimienta 2010), sender-receiver cheap-talk games (Park 1997), zero sum or common interest games (Govindan and McLennan (1998), Litan and Marhuenda (2012)) and games with three players and two strategies each (Litan, Marhuenda, and Sudhölter (2015)).

Clarifying for what types of game forms the number of probability distributions on outcomes induced by the Nash equilibria of the associated game is generically finite remains an open problem. In the present work we address this question and provide a partial answer. We find sufficient and necessary conditions for the generic finiteness of the number of distributions on outcomes, induced by the completely mixed Nash equilibria associated to a bimatrix outcome game form. These are specified in terms of the ranks of two matrices constructed from the original game form and can be checked automatically.

2 Outcome game forms with two players

Let $S^1 = \{1, 2, \ldots, m\}$ and $S^2 = \{1, 2, \ldots, n\}$ be the two players' sets of pure strategies. Let $S = S^1 \times S^2$ and consider a finite set of outcomes Ω. We denote by $\Delta(\Omega)$ (resp. $\Delta_+(\Omega)$) the set of (resp. strictly

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positive) probability measures on Ω. An *outcome game form* is a function $\phi : S \to \Delta(\Omega)$. We regard ϕ as an $m \times n$ matrix For each outcome $\omega \in \Omega$ the mapping ϕ defines a real valued $m \times n$ matrix ϕ^ω whose entry ij is the probability that $\phi(i,j)$ assigns to the outcome $\omega \in \Omega$.

Agents have utilities on outcomes $u \in \mathbb{R}^\Omega$, which canonically extends to $\Delta(\Omega)$. To each $u \in \mathbb{R}^\Omega$ we assign the matrix

$$u(\phi) = \sum_{\omega \in \Omega} u(\omega)\phi^\omega$$

Given two profiles of utilities on outcomes $u^1, u^2 \in \mathbb{R}^\Omega$ for the players, the matrices $u^1(\phi)$ and $u^2(\phi)$ define a two-person game denoted by $\left(u^1(\phi), u^2(\phi)\right)$. A pair of strategies $(x, y) \in \Delta(S^1) \times \Delta(S^2)$ is a *Nash equilibrium* (NE) if $u^1(x, y) \geqslant u^1(i, y)$ and $u^2(x, y) \geqslant u^2(x, j)$ for all $i \in S^1$ and $j \in S^2$. It is a *completely mixed NE* (CMNE) if, in addition, $x \in \Delta_+(S^1)$ and $y \in \Delta_+(S^2)$. The strategies $x \in \Delta(S^1)$ and $y \in \Delta(S^2)$ of the players induce a probability distribution on Ω that assigns the probability $x\phi^\omega y$ to the outcome $\omega \in \Omega$.

We identify $\mathbb{R}^\Omega$ with Euclidean space $\mathbb{R}^{|\Omega|}$. Then, the entries of $u(\phi)$ are linear functions of u. We say that a subset of $\mathbb{R}^\Omega$ is *generic* if it contains an open and dense subset. For $l \in \mathbb{N}$, let d_l denote the vector $(1, \ldots, 1) \in \mathbb{R}^l$.

The number of pure NEs of a finite game is finite. In the case of games with a mixed NE, by eliminating those strategies that are played with zero probability, we will focus on the completely mixed NEs (CMNEs) of the corresponding subgames. Given two utility profiles $u^1, u^2 \in \mathbb{R}^\Omega$, if a pair $(x, y) \in \Delta_+(S^1) \times \Delta_+(S^2)$ of completely mixed strategies is a Nash equilibrium (NE) of the game $\left(u^1(\phi), u^2(\phi)\right)$, then (x, y) is a solution of the following systems of linear equations:

$$u^1(\phi)y = \alpha d_m, \qquad y \cdot d_n = 1 \tag{1}$$

$$xu^2(\phi) = \beta d_n, \qquad x \cdot d_m = 1 \tag{2}$$

for some $\alpha \in \mathbb{R}$ (the payoff of player 1) and $\beta \in \mathbb{R}$ (the payoff of player 2).

Definition 2.1. Given an outcome game form ϕ and a pair of utility profiles $u^1, u^2 \in \mathbb{R}^\Omega$, a *quasi-equilibrium* (QE) of the game $\left(u^1(\phi), u^2(\phi)\right)$ is a pair of solutions $(x, y) = \left(x\left(u^2\right), y\left(u^1\right)\right) \in \mathbb{R}^m \times \mathbb{R}^n$ of the system of equations (1) and (2), for some $\alpha = \alpha\left(u^1\right) \in \mathbb{R}$ and $\beta = \beta\left(u^2\right) \in \mathbb{R}$.

For the rest of the paper we fix an outcome game form ϕ. We rely on the following fact shown by Mas-Colell (2010). Let $k = \max\{\operatorname{rank} u(\phi) : u \in \mathbb{R}^\Omega\}$. There is a generic subset G of $\mathbb{R}^\Omega$ such that the following conditions hold:

(a) $\operatorname{rank} u(\phi) = k$, for every $u \in G$. After reordering, if necessary, the strategies of the players we may write

$$u(\phi) = \begin{pmatrix} B & C \\ D & E \end{pmatrix} \tag{3}$$

where $B = B(u)$ is a $k \times k$ matrix with $\det B \neq 0$.[1] If B is not uniquely determined by the foregoing requirement, we choose one suitable B, with the proviso that the same reordering is applied for all the utilities in the generic G.

(b) The functions $k_1 = \operatorname{rank}\left(u(\phi)|d_m\right)$ and $k_2 = \operatorname{rank}\left(u(\phi)^t|d_n\right)$ are constant on G.

For an $m \times n$ matrix A and $b \in \mathbb{R}^m$, $(A|b)$ is the $m \times (n+1)$ matrix that arises from A by adding b as final column. Note that $k_1, k_2 \in \{k, k+1\}$. Consider the following polynomial on $|\Omega|$ variables,

$$p(u) = \det B\left(u\right)\left(d_k B^{-1}\left(u\right) d_k\right), \quad u \in \mathbb{R}^\Omega \tag{4}$$

Proposition 2.2. *If $k = m = n$, then generically there is at most one QE and, hence, at most one CMNE. Suppose $k < \max\{m, n\}$. Then,*

[1] When there is no danger of confusion, we will not write explicitly the dependence on the utility u for matrices.

(a) If $k < \max\{k_1, k_2\}$, then for every $u^1, u^2 \in G$ the game $\left(u^1(\phi), u^2(\phi)\right)$ has no QE *and, hence, no* CMNE.

(b) If $k_1 = k_2 = k$ and p is the zero polynomial, then for any $u^1, u^2 \in G$ the game $\left(u^1(\phi), u^2(\phi)\right)$ has no QE *and, hence, no* CMNE.

(c) If $k_1 = k_2 = k$ and p is not the zero polynomial, then, for every $u^1, u^2 \in U = \{u \in G : p(u) \neq 0\}$ there is a continuum of QEs *of the game $\left(u^1(\phi), u^2(\phi)\right)$. Furthermore, the systems of linear equations* (1) *and* (2) *have a solution only for the following payoffs*

$$\alpha\left(u^1\right) = \frac{1}{d_k B^{-1}\left(u^1\right) d_k} \text{ and } \beta\left(u^2\right) = \frac{1}{d_k B^{-1}\left(u^2\right) d_k}. \tag{5}$$

Proof. The case $k_1 = k_2 = k$ of the Proposition is a standard result in elementary Linear Algebra. See Litan and Marhuenda (2012) for the proof of parts (b) and (c).

We prove next part (a). We consider only the case $k_1 = k + 1$. The case $k_2 = k + 1$ is similar. We remark first that, since d_m is not a linear combination of the columns of $u(\phi)$, any $y \in \mathbb{R}^n$ which is a solution of (1) for some $u \in G$, must satisfy $u(\phi)y = 0$.

Assume now that there are $u^1, u^2 \in G$ such that the game $\left(u^1(\phi), u^2(\phi)\right)$ has a QE (x, y) with $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$. Then, $u^1(\phi)y = 0$ by the previous remark. Since, G is open, there exists $\varepsilon > 0$ such that $\bar{u} = u^1 + \varepsilon d_{|\Omega|} \in G$. Moreover, (x, y) is also a QE of the game $\left(\bar{u}(\phi), u^2(\phi)\right)$ and $\bar{u}(\phi)\, y = \varepsilon d_m$. But, this contradicts the remark above. Hence, part (a) of the Proposition follows. □

Note that the set U defined in the previous proposition is generic whenever p is not the zero polynomial. Since, we are only interested in the existence of a continuum of CMNEs, from now we suppose the following.

Assumption 2.3.

- *$k_1 = k_2 = k < \max\{m, n\}$.*
- *The polynomial p in* (4) *is not the zero polynomial.*

For $V \subset \mathbb{R}^l$ a linear subspace and $a \in \mathbb{R}^l$ we let $aV = Va = \{a \cdot v : v \in V\}$. Let $u \in U$ and define $K_1(u) = \{z \in \mathbb{R}^n : u(\phi)z = 0\} = \ker u(\phi)$ and $K_2(u) = \{t \in \mathbb{R}^m : tu(\phi) = 0\} = \ker u(\phi)^t$. Since, we are assuming that $k_1 = k_2 = k$, we have that $K_2(u) = \ker u(\phi)^t = \ker\left(u(\phi)^t | d_n\right)$ and $K_1(u) = \ker u(\phi) = \ker\left(u(\phi)|d_m\right)$. It follows that $\dim K_2(u) = m - k$, $\dim K_1(u) = n - k$ and $d_n K_1(u) = d_m K_2(u) = 0$. In addition, any $z \in K_2\left(u^1\right)$ may be written as $z = y_1 - y_2$ with y_1, y_2 two solutions of the system of equations (1) and any $t \in K_1\left(u^1\right)$ may be written as $t = x_1 - x_2$ with x_1, x_2 two solutions of the system of equations (2).

Let $\alpha, \beta : U \to \mathbb{R}$ as in (5). Define the functions $y^p : U \to \mathbb{R}^n$, $y^h : U \times \mathbb{R}^{n-k} \to \mathbb{R}^n$, $x^p : U \to \mathbb{R}^m$ and $x^h : U \times \mathbb{R}^{m-k} \to \mathbb{R}^m$ by

$$\begin{aligned} y^p(u) &= \alpha(u)\left(B^{-1}(u)d_k, 0\right) \\ y^h(u, v) &= \left(-B^{-1}(u)C(u)v, v\right) \\ x^p(u) &= \left(\beta(u)d_k B^{-1}(u), 0\right) \\ x^h(u, w) &= \left(-wD(u)B^{-1}(u), w\right) \end{aligned} \tag{6}$$

The following is proved in Litan and Marhuenda (2012).

Lemma 2.4. *Let Assumption 2.3 hold and let $u \in U$. Then,*

(a) $u(\phi)\, y^p\,(u) = \alpha\,(u)\, d_m, d_n \cdot y^p\,(u) = 1$;

(b) $u(\phi)y^h(u, v) = 0, d_n \cdot y^h(u, v) = 0$, for every $v \in \mathbb{R}^{n-k}$;

(c) $x^p(u)u(\phi) = \beta(u)d_n, d_m \cdot x^p(u) = 1$*;*

(d) $x^h(u,w)\, u(\phi) = 0, d_m \cdot x^h(u,w) = 0$*, for every* $w \in \mathbb{R}^{m-k}$*;*

It follows now from Lemma 2.4 that $K_1(u) = \{y^h(u,v) : v \in \mathbb{R}^{n-k}\}$, $K_2(u) = \{x^h(u,w) : w \in \mathbb{R}^{m-k}\}$ and that every solution $\bar{y}$ of (2) may be written as $\bar{y} = y^p(u^1) + y^h(u^1,v)$ for some $v \in \mathbb{R}^{n-k}$ and every solution $\bar{x}$ of (1) may be written as $\bar{x} = x(u^2) + x^h(u^2,w)$ for some $w \in \mathbb{R}^{m-k}$.

Definition 2.5. Let $u^1, u^2 \in \mathbb{R}^\Omega$. A vector $z \in \mathbb{R}^\Omega$ is a *quasi(probability)-distribution* on Ω for the game $\left(u^1(\phi), u^2(\phi)\right)$ if there exists a QE (x,y) of that game such that for each $\omega \in \Omega$, $z(\omega) = x\phi^\omega y$. Denote

$$O(u^1,u^2) = \{z \in \mathbb{R}^\Omega : z \text{ is a quasi-distribution on } \Omega \text{ for the game } \left(u^1(\phi), u^2(\phi)\right)\}.$$

Note that the set of QE's of $\left(u^1(\phi), u^2(\phi)\right)$ is convex so that $O(u^1,u^2)$ is connected. Let $u^1, u^2 \in U$. Let x and y be a QE. From now on, we follow the notation of Lemma 2.4 and we will write $x = x(u^2,w) = x^p + x^h$, $y = y(u^1,v) = y^p + y^h$, with $x^h = x^h(u^2,w) \in K_2(u^2)$, $y^h = y^h(u^1,v) \in K_1(u^1)$, $y^p = y^p(u^1)$ a particular solution of the system of equations (1) and $x^p = x^p(u^2)$ a particular solution of the system of equations (2). The probability that outcome ω occurs is

$$x\phi^\omega y = x^p\phi^\omega y^p + x^p\phi^\omega y^h + x^h\phi^\omega y^p + x^h\phi^\omega y^h \tag{7}$$

Lemma 2.6. *Suppose that Assumption 2.3 holds. Let* $\omega \in \Omega$*,* $u^1, u^2 \in U$*. If either of the following two conditions hold*

(a) $x\phi^\omega K_1(u^1) = 0$ *for every* $x = x(u^2)$ *a solution of the system of equations* (2)*.*

(b) $K_2(u^2)\phi^\omega y = 0$ *for every* $y = y(u^1)$ *a solution of the system of equations* (1)*.*

then, $K_2(u^2)\phi^\omega K_1(u^1) = 0$*.*

Proof. Suppose that condition (a) holds. Let $t = x_1 - x_2 \in K_2(u^2)$, where x_1, x_2 are two solutions of the system of equations (2). Let $z \in K_1(u^1)$. Then, $t\phi^\omega z = x_1\phi^\omega z - x_2\phi^\omega z = 0$. Hence, it follows that $K_2(u^2)\phi^\omega K_1(u^1) = 0$. Similarly, condition (b) implies that $K_2(u^2)\phi^\omega K_1(u^1) = 0$. □

Proposition 2.7. *Suppose that Assumption 2.3 holds. Let* $u^1, u^2 \in U$*. The set of QE of the game defined by* $u^1(\phi)$ *and* $u^2(\phi)$ *induce finitely many quasi-distributions on* Ω *if and only if the following two conditions hold.*

(a) $x\phi^\omega K_1(u^1) = 0$ *for every* $\omega \in \Omega$ *and every* $x = x(u^2)$ *a solution of the system of equations* (2)*.*

(b) $K_2(u^2)\phi^\omega y = 0$ *for every* $\omega \in \Omega$ *and for every* $y = y(u^1)$ *a solution of the system of equations* (1)*.*

Proof. Since, $O(u^1,u^2)$ is connected, the QEs induce finitely many quasi-distributions on outcomes if and only they induce a unique quasi-distribution on outcomes.

Assume there is a unique quasi-distribution induced on outcomes by the QEs of the game. Let $x = x(u^2)$ be a solution of (2), and $z = z(u^1) \in K_1(u^1)$. We write $z = y_1 - y_2$, where y_1, y_2 are two solutions of (1). Then, for each $\omega \in \Omega$, we have that $x\phi^\omega y_1 = x\phi^\omega y_2$. So,

$$x\phi^\omega z = x\phi^\omega y_1 - x\phi^\omega y_2 = 0$$

Thus, $x\phi^\omega K_1(u^1) = 0$ and (a) follows. Similarly, we can prove (b).

Conversely, suppose that conditions (a) and (b) hold. Let $u^1, u^2 \in U$ and $\omega \in \Omega$. By Lemma 2.6 we have that $K_2(u^2)\phi^\omega K_1(u^1) = 0$. Therefore, in (7), for every QE (x,y) we have that $x\phi^\omega y = x^p\phi^\omega y^p$. It follows that the QEs of the game defined by $u^1(\phi)$ and $u^2(\phi)$ induce a unique quasi-distribution on Ω. □

Corollary 2.8. *Let Assumption 2.3 hold. Suppose there are two sets of vectors $V_1 \subset \mathbb{R}^n$ and $V_2 \subset \mathbb{R}^m$ such that for every $u \in U$ and $i = 1, 2$ we have that V_i generates $K_i(u)$. Then, for any $u^1, u^2 \in U$ the set of CMNE of the game $\left(u^1(\phi), u^2(\phi)\right)$ induce finitely many probability distributions on outcomes.*

Proof. Let $\omega_i \in \Omega$, $z \in V_1$ and $t \in V_2$. For each $u \in U$, we have that $tu(\phi) = u(\phi)z = 0$. Differentiating with respect to u_i we see that $t\phi^{\omega_i} = \phi^{\omega_i} z = 0$ for every $z \in V_1$ and $t \in V_2$. Hence, $\phi^{\omega_i} K_1(u) = K_2(u)\phi^{\omega_i} = 0$ and the result follows from Proposition 2.7. □

Corollary 2.9. *Let Assumption 2.3 hold and suppose $k = 1, 2$. Then, for any $u^1, u^2 \in U$ the set of CMNE of the game $\left(u^1(\phi), u^2(\phi)\right)$ induce finitely many probability distributions on outcomes.*

Proof. We show the result for $k = 2$. Let $u^1, u^2 \in U$. Write the entries of ϕ as $\phi(i,j) = \sum_{\omega\in\Omega} \phi^\omega(i,j)\omega$, with $\sum_{\omega\in\Omega} \phi^\omega(i,j) = 1$. Then, the ij entry of matrix $u^1(\phi)$ is $\sum_{\omega\in\Omega} \phi^\omega(i,j)u^1(\omega)$. By assumption, each of the columns $3, \ldots, n$ of ϕ is a linear combination of the columns 1 and 2 of ϕ. That is for $j = 3, \ldots, n$ we have that $\phi(i,j) = \lambda_{1j}\phi(i,1) + \lambda_{2j}\phi(i,2)$, for some $\lambda_{1j}, \lambda_{2j}$. Hence,

$$\sum_{\omega\in\Omega} \phi^\omega(i,j)\omega = \sum_{\omega\in\Omega} \left(\lambda_{1j}\phi^\omega(i,1) + \lambda_{2j}\phi^\omega(i,2)\right)\omega$$

Therefore, $\phi^\omega(i,j) = \lambda_{1j}\phi^\omega(i,1) + \lambda_{2j}\phi^\omega(i,2)$, for every $\omega \in \Omega$. Adding for $\omega \in \Omega$, we obtain that $\lambda_{1j} + \lambda_{2j} = 1$. Since, the first two columns of ϕ are linearly independent, there must be some $i_0 = 1, \ldots, n$ such that $\phi^\omega(i_0, 1) \neq \phi^\omega(i_0, 2)$. Then,

$$\begin{aligned} \lambda_{1j} &= \frac{\phi^\omega(i_0,j) - \phi^\omega(i_0,2)}{\phi^\omega(i_0,1) - \phi^\omega(i_0,2)} \in \mathbb{R} \qquad j = 3, \ldots, n \\ \lambda_{2j} &= 1 - \lambda_{1j} \in \mathbb{R} \end{aligned}$$

are independent of u^1. Hence, we have shown that the set of vectors

$$V_1 = \{(\lambda_{13}, \lambda_{23}, -1, 0, \ldots, 0), (\lambda_{14}, \lambda_{24}, 0, -1, 0, \ldots, 0), \ldots (\lambda_{1n}, \lambda_{2n}, 0, \ldots, 0, -1)\} \subset \mathbb{R}^n$$

is a basis of $K_1(u^1)$ which is independent of u^1. In a similar manner, we can find a set of vectors $V_2 \subset \mathbb{R}^m$ that generates $K_2(u^2)$. The result follows now from Corollary 2.8. □

As an application, we see that the following result already obtained in Litan, Marhuenda, and Sudhölter (2015) now follows from Proposition 2.2 (a) and Corollary 2.9.

Corollary 2.10. *Suppose* $\max\{m, n\} \leq 3$. *Then, for any $u^1, u^2 \in U$ the set of CMNE of the game $\left(u^1(\phi), u^2(\phi)\right)$ induce finitely many probability distributions on outcomes.*

The following result provides sufficient and necessary conditions for the existence of CMNEs in games induced by outcome game forms.

Theorem 2.11. *Let Assumption 2.3 hold, $u^1, u^2 \in U$ and suppose there is, at least, one CMNE of the game $\left(u^1(\phi), u^2(\phi)\right)$. Then, all the CMNE of that game induce finitely many probability distributions on outcomes iff for every $\omega \in \Omega$*

$$\text{rank}\begin{pmatrix} u^1(\phi) & 0 \\ \phi^\omega & u^2(\phi) \\ 0 & d_n \end{pmatrix} = 2k, \quad \text{rank}\begin{pmatrix} u^2(\phi) & \phi^\omega & 0 \\ 0 & u^1(\phi) & d_m \end{pmatrix} = 2k$$

The proof is based on techniques developed in Litan and Marhuenda (2012). A review of these and the details of the proof are provided in the appendix.

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A The proof of Theorem 2.11

Lemma A.1. *Let $\omega \in \Omega$. Let Assumption 2.3 and either of the following two conditions hold.*

(a) $x\phi^\omega K_1\left(u^1\right) = 0$ for every $x = x\left(u^2\right)$ a solution of the system of equations (2).

(b) $K_2(u^2)\phi^\omega y = 0$ for every $y = y\left(u^1\right)$ a solution of the system of equations (1).

Then for any $u^1, u^2 \in U$ we have that

$$\operatorname{rank}\begin{pmatrix} u^1(\phi) & 0 \\ \phi^\omega & u^2(\phi) \end{pmatrix} = 2k$$

Proof. Let $u^1, u^2 \in U$ and

$$F = \begin{pmatrix} u^1(\phi) & 0 \\ \phi^\omega & u^2(\phi) \end{pmatrix}$$

We use the notation

$$\phi^\omega = \begin{pmatrix} B^\omega & C^\omega \\ D^\omega & E^\omega \end{pmatrix}$$

to denote the decomposition of the matrix A in (3) applied to the matrix ϕ^ω. We can write now

$$F=\begin{pmatrix} B\left(u^1\right) & C\left(u^1\right) & 0 & 0 \\ D\left(u^1\right) & E\left(u^1\right) & 0 & 0 \\ B^\omega & C^\omega & B\left(u^2\right) & C\left(u^2\right) \\ D^\omega & E^\omega & D\left(u^2\right) & E\left(u^2\right) \end{pmatrix}$$

By elementary row and column operations,

$$\operatorname{rank} F=\operatorname{rank}\begin{pmatrix} B\left(u^1\right) & C\left(u^1\right) & 0 \\ B^\omega & C^\omega & B\left(u^2\right) \\ D^\omega & E^\omega & D\left(u^2\right) \end{pmatrix}=\operatorname{rank}\begin{pmatrix} B\left(u^1\right) & C\left(u^1\right) & 0 \\ B^\omega & C^\omega & B\left(u^2\right) \\ D_1^\omega & E_1^\omega & 0 \end{pmatrix}$$

where

$$D_1^\omega = D^\omega - D\left(u^2\right)B^{-1}\left(u^2\right)B^\omega \quad (8)$$
$$E_1^\omega = E^\omega - D\left(u^2\right)B^{-1}\left(u^2\right)C^\omega \quad (9)$$

Finally,

$$\operatorname{rank} F=\operatorname{rank}\begin{pmatrix} B\left(u^1\right) & 0 & 0 \\ B^\omega & C_2^\omega & B\left(u^2\right) \\ D_1^\omega & E_2^\omega & 0 \end{pmatrix}$$

with $C_2^\omega = C^\omega - B^\omega B^{-1}\left(u^1\right)C\left(u^1\right)$ and

$$\begin{aligned} E_2^\omega &= E^\omega - D\left(u^2\right)B^{-1}\left(u^2\right)C^\omega - D^\omega B^{-1}\left(u^1\right)C\left(u^1\right) + D\left(u^2\right)B^{-1}\left(u^2\right)B^\omega B^{-1}\left(u^1\right)C\left(u^1\right) \\ &= \begin{pmatrix} -D\left(u^2\right)B^{-1}\left(u^2\right) & I_{m-k} \end{pmatrix}\begin{pmatrix} B^\omega & C^\omega \\ D^\omega & E^\omega \end{pmatrix}\begin{pmatrix} -B^{-1}\left(u^1\right)C\left(u^1\right) \\ I_{n-k} \end{pmatrix} \\ &= \begin{pmatrix} -D\left(u^2\right)B^{-1}\left(u^2\right) & I_{m-k} \end{pmatrix}\phi^\omega\begin{pmatrix} -B^{-1}\left(u^1\right)C\left(u^1\right) \\ I_{n-k} \end{pmatrix} \end{aligned}$$

Now, by Lemma 2.6 we have $K_2(u^2)\phi^\omega K_1(u^1)=0$. Hence, for any $v\in\mathbb{R}^{n-k}$ and $w\in\mathbb{R}^{m-k}$ we have

$$0=w\begin{pmatrix} -D\left(u^2\right)B^{-1}\left(u^2\right) & I_{m-k} \end{pmatrix}\phi^\omega\begin{pmatrix} -B^{-1}\left(u^1\right)C\left(u^1\right) \\ I_{n-k} \end{pmatrix}v$$

Therefore, $E_2^\omega=0$ and

$$\operatorname{rank} F=\operatorname{rank}\begin{pmatrix} B\left(u^1\right) & 0 & 0 \\ B^\omega & C_2^\omega & B\left(u^2\right) \\ D_1^\omega & 0 & 0 \end{pmatrix}=\operatorname{rank}\begin{pmatrix} B\left(u^1\right) & 0 & 0 \\ B^\omega & C_2^\omega & B\left(u^2\right) \end{pmatrix}=2k$$

because $\operatorname{rank} B\left(u^1\right)=\operatorname{rank} B\left(u^2\right)=k$. □

The following follows now from basic Linear Algebra.

Lemma A.2. *Suppose that Assumption 2.3 holds. Let $u^1,u^2\in U$. The set of QE of the game $\left(u^1(\phi),u^2(\phi)\right)$ induce finitely many quasi-distributions on outcomes if and only if for every $\omega\in\Omega$ and every QE $\left(x\left(u^2\right),y\left(u^1\right)\right)$ of the above game we have that*

(a) $x\left(u^2\right)\phi^\omega$ *is in the image of* $u^1(\phi)^t$.

(b) $\phi^\omega y\left(u^1\right)$ *is in the image of* $u^2(\phi)$.

Lemma A.3. *Suppose that Assumption 2.3 holds. Let* $u^1, u^2 \in U$, $\omega \in \Omega$. *The following hold.*

(a) $x\phi^\omega K_1\left(u^1\right) = 0$ *for every solution* $x = x\left(u^2\right)$ *of the system of equations* (2) *if and only if*

$$\operatorname{rank}\begin{pmatrix} u^1(\phi) & 0 \\ \phi^\omega & u^2(\phi) \\ 0 & d_n \end{pmatrix} = 2k$$

(b) $K_2(u^2)\phi^\omega y = 0$ *for every solution* $y = y\left(u^1\right)$ *of the system of equations* (1) *if and only if*

$$\operatorname{rank}\begin{pmatrix} u^2(\phi) & \phi^\omega & 0 \\ 0 & u^1(\phi) & d_m \end{pmatrix} = 2k$$

Proof. We prove only part (a). The proof of part (b) is similar. Fix a solution x of the system of equations (2). Let

$$F = \begin{pmatrix} u^1(\phi) & 0 \\ \phi^\omega & u^2(\phi) \\ 0 & d_n \end{pmatrix}$$

Since, $d_n = \frac{1}{\beta(u^2)} x u^2(\phi)$, we have that

$$\operatorname{rank} F = \operatorname{rank}\begin{pmatrix} u^1(\phi) & 0 \\ \phi^\omega & u^2(\phi) \\ \frac{1}{\beta(u^2)} x\phi^\omega & 0 \end{pmatrix}$$

Assume that $x\phi^\omega K_1\left(u^1\right) = 0$. Then, by Lemma A.2, $x\phi^\omega$ is in the image of $u^1(\phi)^t$ and hence a linear combination of the rows of $u^1(\phi)$. Hence,

$$\operatorname{rank} F = \operatorname{rank}\begin{pmatrix} u^1(\phi) & 0 \\ \phi^\omega & u^2(\phi) \end{pmatrix}$$

and, by Lemma A.1 we have that $\operatorname{rank} F = 2k$.

Conversely, suppose now that $\operatorname{rank} F = 2k$. We proceed now as in Lemma A.1 and write F as

$$F = \begin{pmatrix} \multicolumn{2}{c}{u^1(\phi)} & 0 & 0 \\ B^\omega & C^\omega & B\left(u^2\right) & C\left(u^2\right) \\ D^\omega & E^\omega & D\left(u^2\right) & E\left(u^2\right) \\ \multicolumn{2}{c}{\frac{1}{\beta(u^2)} x(u^2)\phi^\omega} & 0 & 0 \end{pmatrix}$$

Since $\operatorname{rank} u^2(\phi) = \operatorname{rank} B\left(u^2\right) = k$, we get that

$$\operatorname{rank} F = \operatorname{rank}\begin{pmatrix} \multicolumn{2}{c}{u^1(\phi)} & 0 \\ B_1^\omega & C_1^\omega & B\left(u^2\right) \\ D_1^\omega & E_1^\omega & 0 \\ \multicolumn{2}{c}{\frac{1}{\beta(u^2)} x\phi^\omega} & 0 \end{pmatrix} = 2k$$

where D_1^ω, E_1^ω are defined in (8) and (9). Since, $\operatorname{rank} u^1(\phi) = \operatorname{rank} B\left(u^2\right) = k$, the rows of the matrix $\begin{pmatrix} D_1^\omega & E_1^\omega \end{pmatrix}$ and $x\phi^\omega$ are a linear combination of the rows of $u^1(\phi)$. Hence, (a) follows. $\square$

The following result follows now immediately from Proposition 2.7 and Lemma A.3.

Theorem A.4. *Suppose that Assumption 2.3 holds. Let $u^1, u^2 \in U$. Then, all the QE of the game $\left(u^1(\phi), u^2(\phi)\right)$ induce finitely many quasi-distributions on outcomes iff for every $\omega \in \Omega$*

$$\operatorname{rank}\begin{pmatrix} u^1(\phi) & 0 \\ \phi^\omega & u^2(\phi) \\ 0 & d_n \end{pmatrix} = 2k, \quad \operatorname{rank}\begin{pmatrix} u^2(\phi) & \phi^\omega & 0 \\ 0 & u^1(\phi) & d_m \end{pmatrix} = 2k$$

Thus, under Assumption 2.3, if the rank conditions in Theorem A.4 hold, the set of QE induce a unique quasi-distribution on outcomes. Since the set of CMNE is a subset of the set of QE, the set of CMNE also induces, at most, a unique distribution on outcomes. Thus, the 'if' part of Theorem 2.11 holds.

Conversely, let $u^1, u^2 \in U$ and let Assumption 2.3 hold. Suppose that the game $\left(u^1(\phi), u^2(\phi)\right)$ has, at least, a CMNE , say $\bar{x} = x^p(u^2) + x^h(u^2, w_0) \in \Delta_+(S^1)$ a solution of (2) and $\bar{y} = y^p\left(u^1\right) + y^h(u^1, v_0) \in \Delta_+(S^2)$ a solution of (1), with $w_0 \in \mathbb{R}^{m-k}, v_0 \in \mathbb{R}^{n-k}$. And suppose that the set of CMNE of the game $\left(u^1(\phi), u^2(\phi)\right)$ induces finitely many distributions on outcomes. By continuity, for $v \in \mathbb{R}^{n-k}$ close enough to v_0 and $w \in \mathbb{R}^{m-k}$ close enough to w_0, we have that $x = x^p(u^2) + x^h(u^2, w) \in \Delta_+(S^1)$ and $y = y^p\left(u^1\right) + y^h(u^1, v) \in \Delta_+(S^2)$ is a CMNE of the game $\left(u^1(\phi), u^2(\phi)\right)$.

Suppose now that the rank conditions in Theorem A.4 do not hold. Then for some outcome $\omega \in \Omega$ the function

$$q_\omega(v, w) = \left(x^p(u^2) + x^h(u^2, w)\right)\phi^\omega\left(y^p\left(u^1\right) + y^h(u^1, v)\right)$$

takes a continuum of values. Since, it is a polynomial in $v \in \mathbb{R}^{n-k}$ and $w \in \mathbb{R}^{m-k}$ it also takes a continuum of values on any open set of $\mathbb{R}^{n-k} \times \mathbb{R}^{m-k}$. Hence, the set CMNE of the game $\left(u^1(\phi), u^2(\phi)\right)$ do not induce finitely many distributions on outcomes, which contradicts our assumption and the 'only if' part of Theorem 2.11 follows.