Deficit Rules and Monetization in a Growth Model with Multiplicity and Indeterminacy

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Abstract

This paper develops a formal analysis of deficit monetization in a monetary endogenous growth model based on transaction costs, in which economic growth interacts with productive public expenditures. This interaction generates two positive balanced growth paths (BGP) in the long-run: a high BGP and a low BGP. Transitional dynamics shows that multiplicity cannot be rejected if transaction costs affect both consumption and investment expenditures, with possible indeterminacy of the high BGP. Deficit monetization is shown to reduce the parameters-space producing indeterminacy.

1. Introduction

The Great Recession shaped the conduct of monetary and fiscal policies in an unprecedented way. On the fiscal side, Governments of developed countries launched massive debt-financed spending programs that might have generated potentially explosive debt paths. On the monetary side, many Central Banks implemented "unconventional" monetary policies, and bought an unprecedented amount of public (and private) debt, sometimes even despite institutional arrangements prohibiting monetizing sovereign debts (like in the Eurozone). Consequently, the age of Central Banks "independence" (from fiscal policy) and of monetary policy isolationism seems to be over (see, e.g. Taylor, 2012).

From the theoretical perspective, the impact of monetizing public debts and deficits on inflation is a long-standing question, since the seminal "*unpleasant monetarist arithmetics*" of Sargent and Wallace (1981) and its developments through the "fiscal theory of the price level" (see Aiyagari and Gertler, 1985; Leeper, 1991; Woodford, 1994). However, to the best of our knowledge, there is no work that addresses the question of the impact of deficit monetization on economic growth, despite of its relevance, nowadays and from a historical perspective (see e.g. Rousseau and Stroup (2011)).[1]

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[1] Some recent contributions study the multiplier effect of money-financed public expenditures in the short-run (see, e.g. Gali (2014) and Buiter (2014)), without addressing the question of deficit monetization.

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In this paper, to correct for this caveat, we build an endogenous growth model that allows for permanent public indebtedness, in order to study the growth impact of monetizing public deficits. To give a role for public expenditures, we model endogenous growth from the canonical model of Barro (1990) with public spending entering the production function as a flow of productive services. In this model, we introduce several innovations. First, to deal with the question of public debt and deficit monetization, we consider a general budget constraint for the Government, in which public expenditures can be financed by taxes, public debt or money emissions. This creates a richer environment to study Government finance, compared to the balanced budget rule used by Barro (1990). Second, contrary to usual modeling of an exogenous money supply, we suppose that money creation is proportional to fiscal deficits. This allows analyzing the impact of the degree of deficit-monetization, which in the long-run corresponds to monetizing a share of public debt. Third, to introduce money, we resort to a generalized transaction costs specification based on the fact that resources are used up in the process of exchange (Brock, 1974, 1990). This specification is more general than usual "cash-in-advance' (CIA) models, because it allows for an interest-elastic money demand. Furthermore, it is also more general than "money-in-the-utility-function" (MIUF) approaches, because the demand for money is generated by the need of a liquid asset to finance either consumption goods only or both consumption and investment goods.

Our findings are threefold. First, as regards the balanced growth path (hereafter BGP), our model exhibits a multiplicity of steady-states, namely, a high BGP and a low BGP, due to the dual positive interaction between economic growth and public expenditure. Indeed, on the one hand, the rate of economic growth positively depends on public expenditure, which increases the marginal productivity of private capital. On the other hand, public expenditure is an increasing function of economic growth in the Government budget constraint, because growth allows reducing the debt burden in the long-run. Thus, high economic growth allows to reduce the impact of the debt burden, and boosts growth-enhancing productive expenditure, while low economic growth gives rise to an increase in public debt, with an associated crowding-out effect on productive public spending, which, in turn, has an adverse effect on growth.

Second, as regards the impact of deficit and monetization on the two BGPs, our model shows that the low BGP depends positively on deficits and negatively on monetization. Along the high BGP, however, public deficits increase economic growth only if they are sufficiently monetized (i.e. if monetization is "high"), and the direct impact of monetization depends on the interest-elasticity of the demand for money.

Third, as regards transitional dynamics, results change dramatically depending on what expenditures are subject to the transaction costs. In the special case with transaction costs on consumption only, the high BGP is saddle-point stable and the low BGP is unstable, so that multiplicity can be removed. However, in the general case with transaction costs on both consumption and investment, the low BGP becomes saddle-point stable, and the high BGP becomes locally undetermined or saddle-point stable depending on parameters. Therefore, multiplicity can no longer be excluded: depending on the initial level of the public debt ratio, both steady-states are reachable. If the initial public debt ratio is "high", the economy is condemned to remain in the near context of a poverty trap with economic growth approaching to zero. If the initial public debt ratio is "low", on the contrary, the economy will converge towards the high BGP, but the exact transition path may be undetermined. We show in particular that "high" levels of monetization are beneficial to the determinacy of the high BGP.

Our model can be seen as unifying two strands of literature. On the one hand, it extends and challenges preceding results about the impact of public debt on long-run economic growth. In endogenous growth settings with wasteful public expenditures, Saint-Paul (1992) and Futagami and Shibata (1998) find that higher debt and deficits are harmful to economic growth. These findings have been extended by Minea and Villieu (2010, 2012) in endogenous growth models with productive public expenditures. They show in particular that, even if deficits are devoted to productive expenditure, long-run economic growth is worsened by the presence of public debt, because the crowding-out effect of the debt burden always outmatches the increase in public spending authorized by the deficits along the BGP. The present paper shows that these results can be reversed if deficit are monetized: a sufficiently high dose of monetization would allow to overcome the crowding-out of public debt in the long-run.

On the other hand, an important strand of literature explore the possibility of having money as a source of indeterminacy in endogenous growth models. It is well-known that indeterminacy can arise when the Central Bank follows an exogenous money growth rule (see Michener and Ravikumar, 1998). Indeed, capitalizing on the works of Wang and Yip (1992), Palivos et al. (1993), and Palivos and Yip (1995), who develop CIA endogenous growth models, a large literature emphasized several mechanisms through which the CIA constraint may give rise to multiplicity and/or indeterminacy of BGPs.[2] For example, in an endogenous growth model with transaction costs and endogenous labor supply, Itaya and Mino (2003) show that labor externalities can produce indeterminacy. In Suen and Yip (2005), using an “Ak” model with a CIA money demand, indeterminacy is caused by a strong intertemporal substitution effect on capital accumulation. Moreover, when the CIA only partially affects consumption, Bosi and Magris (2003), in a one-sector, and Bosi et al. (2010), in a two-sector economy, show that indeterminacy and multiplicity can occur. Finally, Bosi and Dufourt (2008) and Chen and Guo (2008) highlight that the form of the CIA constraint, and specifically the extent to which it affects investment, is a key factor in generating indeterminacy. By studying debt and deficits monetization, our model extends these works in several dimensions. First, we introduce a role for government spending, through productive services of public expenditures.[3] Second, we relax an important assumption of this literature, namely the presence of a balanced budget rule. Third, by accounting for the possibility of deficit monetization, we go beyond the hypothesis that money supply is exogenous and study indeterminacy in the context of a “passive” monetary authority, with the degree of monetization being used as a selection device among different convergent paths, since indeterminacy can be removed for high degrees of monetization.

The paper is structured as follows. Section 2 presents the model, Section 3 describes the long-run solution and studies the effect of deficit and monetization along the BGPs. Section 4 discusses transitional dynamics and the way indeterminacy can occur or not depending on parameters. Section 5 concludes the paper.

[2]The possibility of indeterminacy in dynamic equilibrium models has been largely explored in the literature (for a survey of the mechanisms that can give rise to indeterminacy, see, e.g. Benhabib and Farmer (1999)).

[3]In a setup with no debt nor seigniorage, Palivos et al. (2003) and Park and Philippopoulos (2004) show that endogenous public investment can lead to both multiplicity and indeterminacy of BGPs.

2. The model

We consider a continuous-time endogenous growth model describing a closed economy populated by a private sector and fiscal and monetary authorities.

2.1. The private sector

The private sector consists of a producer-consumer infinitely-lived representative agent with perfect foresight, who maximizes the present value of a discounted sum of instantaneous utility functions based on consumption $c_t > 0$. With $\rho > 0$ the discount rate and $S := -u_{cc}c_t/u_c > 0$ (with $u_c := du(c_t)/dc_t$) the consumption elasticity of substitution, Households welfare is

$$U = \int_0^\infty u(c_t)\exp(-\rho t)dt, u(c_t) = \begin{cases} \frac{S}{S-1}\left\{(c_t)^{\frac{S-1}{S}} - 1\right\}, & for\ S \neq 1 \\ \log(c_t), & for\ \ S = 1. \end{cases} \tag{1}$$

For lifetime utility U to be bounded, it must be true that $(S-1)\gamma_c < S\rho$, where γ_x is the long-run growth rate of variable x.[4]

Output is produced with private capital and productive public expenditure g_t

$$y_t = Ak_t^\alpha g_t^{1-\alpha}. \tag{2}$$

All variables are per capita with population normalized to unity. The elasticity of output to private capital is $\alpha \in (0,1)$. Following Barro (1990), public expenditure provides "productive services", with an elasticity $1-\alpha$.

To motivate a demand for real balances, we have to specify some imperfections in the process of exchange, due to "transactions costs". There has been much discussion about the way to introduce money in general equilibrium setups, leading to two alternative usual reduced forms of money demand, namely money in the utility function (MIUF) or "cash-in-advance" (CIA) models. In some sense, the former approach may be viewed as more general, because it gives rise to an interest-elastic demand for money, while, in deterministic setups, the CIA approach leads to a strict quantitative equation with constant velocity of money as soon as the nominal interest rate is positive.[5] Furthermore, the CIA specification is a special case of MIUF, when money and consumption are strict complements in utility (Asako, 1983).[6]

However, this advantage in terms of generality applies if consumption only is subject to the CIA constraint, but not when this constraint affects both investment and consumption goods. Indeed, it would be quite unusual to introduce investment in the utility function. On this ground, the CIA version might be seen as more general. Besides, CIA specifications have been proved to be very sensitive to the type of goods subject to the money constraint. Stockman (1981) first shows that, in a "neoclassical" growth model,

[4] Defining by r the real interest rate, this condition corresponds to the no-Ponzi game constraint $\gamma_c < r$.

[5] Except in models with "cash goods" and "credit goods", see Lucas and Stokey (1987).

[6] Feenstra (1986) extends this result to a large class of models with transaction costs. Nevertheless, this "functional equivalence" does not apply to models with transaction costs on investment.

money is not superneutral in the long-run when the CIA constraint affects both investment and consumption. In addition, as we have stated in the Introduction, multiplicity and indeterminacy depend crucially on the form of the CIA constraint.

In this paper we provide a more general approach that encompasses the above cases. To this end, we develop a transaction costs demand for money, based on the fact that resources are used up in the process of exchange (Brock, 1974, 1990; Jha et al., 2002). We suppose that some expenditures (e_t), to be defined below, are subject to a transaction cost $T(e_t, m_t) = \psi(e_t, m_t)\, e_t$, namely, using an isoelastic specification

$$\psi(e_t, m_t) := \frac{\omega}{1+\mu}(e_t/m_t)^{\mu}, \tag{3}$$

with ω a positive scale parameter ensuring small transaction costs. Function (3) expresses that a fraction of expenditures is lost in the process of exchange. This fraction negatively depends on real balances, since money provides liquidity services. Such a transaction technology gives rise to the CIA constraint: $e_t = m_t$ as a special case when $\mu \to \infty$,[7] but yields an interest-elastic money demand if $\mu < \infty$, which is crucial to our analysis. Furthermore, by defining $e_t = \phi^c c_t + \phi^k(\dot{k}_t + \delta k_t)$, relation (3) allows studying different special cases according to the expenditures that are subject to the transaction costs: transaction costs on consumption goods only ($\phi^c > 0$ and $\phi^k = 0$) or on consumption plus investment ($\phi^c > 0$ and $\phi^k > 0$).

With transaction costs, Households budget constraint is (we define $\dot{x}_t := dx_t/dt, \forall x_t$)

$$\dot{k}_t + \dot{b}_t + \dot{m}_t = r_t b_t + (1-\tau)\, y_t - c_t - \delta k_t - \pi_t m_t - T(e_t, m_t) + l_t. \tag{4}$$

Households use their net income ($(1-\tau)\, y_t$, with τ a flat tax rate on output) to consume (c_t) and to invest ($\dot{k}_t + \delta k_t$, with δ the rate of private capital depreciation). In addition, they can buy Government bonds (b_t), which return the real interest rate r_t, and hold money. All variables are defined in real terms (i.e. deflated by the price level) and $\pi_t m_t$ represents the "inflation tax" on real money holdings. In addition, since goods must be used up in transacting, the households budget constraint must contain the transaction cost term $T(\cdot)$. Finally, to close the model, l_t is a lump-sum transfer that, in equilibrium will be equal to the value of the transaction costs $T(\cdot)$ levied on households.

2.2. Monetary and fiscal authorities

The Government provides productive public expenditures, levies income taxes, and borrows from Households. He also collects the inflation tax on real balances.[8] Hence the budget constraint is, in real terms

$$\dot{b}_t + \frac{\dot{M}_t}{P_t} = r_t b_t + g_t - \tau y_t =: d_t. \tag{5}$$

[7] Writing $m_t = \left[\frac{\theta y_t}{T(y_t, m_t)}\frac{1}{\mu}\right]^{1/\mu} e_t$, we use $\lim_{\mu\to\infty}\left(\frac{1}{\mu}\right)^{1/\mu} \to 1$, thus $m_t \to e_t$.

[8] In our model, high-powered money is the only form of money, so that the Central Bank collects the inflation tax and transfers it to the Government. We ignore possible developments related to the presence of banking and financial sectors.

The budget constraint (5) is an extension of those in Barro (1990) and Minea and Villieu (2012). Barro (1990) considers only balanced-budget-rules ($g_t = \tau y_t$), while Minea and Villieu (2012) introduce public debt, but without money ($\dot{b}_t = r_t b_t + g_t - \tau y_t$). In our model, by using public debt and seigniorage, the Government can make productive expenditure eventually higher than fiscal revenues τy_t. Thus, we define the deficit as d_t. This deficit can be financed either by issuing debt ($\dot{b}_t$) or by issuing money ($\dot{M}_t/P_t$), with M_t and P_t the money stock and the price level, respectively.

To close the model, we have to specify the instruments available for public finance. First, it must be emphasized that, to obtain an endogenous growth solution, productive public expenditure must be endogenous in the Government budget constraint.[9] In what follows, we suppose that the Government adopts a deficit rule, which specifies a gradual adjustment path of the deficit-to-output ratio to a long-run target. Let $d_{yt} := d_t/y_t$ be the deficit-to-GDP ratio and $\theta := d^*/y^*$ its long-run target, where a star denotes steady-state values. At each period, the deficit ratio evolves according to

$$\dot{d}_{yt} = -\xi \left(d_{yt} - \theta\right). \tag{6}$$

Thus, the fiscal policy instruments are the flat tax rate (τ), the target for the deficit-to-GDP ratio in the long-run (θ), and the speed of adjustment of current deficit to this target (ξ). A low value of the last parameter describes a "gradualist" strategy (i.e. the speed of adjustment of the deficit ratio is small), and a high value accounts for a "shock therapy" strategy, which gives rise to a faster reduction in the deficit ratio. Besides, monetary authorities must decide the deficit share they accept to monetize. For simplicity, we assume that a fraction $\eta \in [0,1]$ of the deficit is monetized at each instant,[10] namely, anticipating on money equilibrium (i.e. $M_t/P_t = m_t$)

$$\dot{M}_t/P_t = \dot{m}_t + \pi_t m_t = \eta d_t. \tag{7}$$

It follows that the Government must cover the remaining part of deficit by issuing public debt

$$\dot{b}_t = (1-\eta) d_t. \tag{8}$$

2.3. Equilibrium

By solving Household's program (see Appendix A) we obtain the two following relations

$$\frac{\dot{c}_t}{c_t} = S\left[r_t - \rho - \frac{\phi^c \dot{Q}_t}{1+\phi^c Q_t}\right], \tag{9}$$

$$\frac{(1-\tau)\,\alpha A\left(g_t/k_t\right)^{1-\alpha}}{1+\phi^k Q_t} - \delta = r_t - \frac{\phi^k \dot{Q}_t}{1+\phi^k Q_t}, \tag{10}$$

where $Q_t := \left[\left(\frac{1+\mu}{\mu}\right)\omega^{1/\mu} R_t\right]^{\mu/(1+\mu)}$ is a transaction cost factor that depends on the nominal interest rate (R_t), with $Q_t = R_t$ in the CIA case ($\mu \to \infty$).

[9] In contrast with Nishimura et al. (2015a,b), we consider that productive public expenditures and not the tax rate adjust at each period to fulfill the government budget constraint.

[10] We could introduce an exogenous trend for money supply, without any change in qualitative results.

Relation (9) is the usual Keynes-Ramsey rule obtained in standard optimal growth problems. With transaction costs on consumption goods ($\phi^c > 0$), the consumption path is affected by the nominal interest rate, which represents a part of the effective cost of consumption. Thus, in periods with increasing (decreasing) nominal interest rates, the growth rate of consumption will be lower (higher) than under the usual Keynes-Ramsey rule. Relation (10) defines the real return of capital. In the absence of transaction costs on capital goods, this return is simply the real interest rate (the rate of return of Government bonds). With transaction costs on investment $\left(\phi^k > 0\right)$, the return of capital is lower, since it must be deflated by the financing cost $\left(1 + \phi^k Q_t\right)$, as shown by first term in the LHS of (10). In addition, the nominal interest rate introduces a wedge between the return of bonds and the return of capital: with a growing nominal interest rate, the return of capital will be lower, as shown by the second term of the RHS of (10).

Since we are interested in endogenous growth solutions, we transform variables into long-run stationary ratios. To do this, we deflate all steady-state growing variables by the capital stock, namely $x_k := x_t/k_t$ (and we henceforth remove time indexes). Thus, the path of the capital stock is obtained from the goods market equilibrium

$$\frac{\dot{k}}{k} = y_k - c_k - g_k - \delta, \tag{11}$$

with the production function defined as

$$y_k = A g_k^{1-\alpha}, \tag{12}$$

and the demand for real balances is (see Appendix A)

$$m_k = e_k \left(Q_t/\omega\right)^{-1/\mu}, \tag{13}$$

where, using (11),

$$e_k = \phi^k \left(A g_k^{1-\alpha} - g_k\right) + \left(\phi^c - \phi^k\right) c_k. \tag{14}$$

Observe that money demand is interest-elastic, except in the CIA special case. We then extract the deficit-to-capital ratio from the Government budget constraint (5)

$$d_k = r b_k + g_k - \tau y_k, \tag{15}$$

and the behavior of monetary and fiscal authorities (7)-(8) leads to

$$\frac{\dot{m}}{m} = \eta \left(\frac{d_k}{m_k}\right) - \pi, \tag{16}$$

$$\frac{\dot{b}}{b} = (1 - \eta) \left(\frac{d_k}{b_k}\right). \tag{17}$$

Assuming Fisher equation $R = r + \pi$, relations (9)-(17), together with the deficit rule (6), fully characterize the equilibrium of the model.

3. The long-run endogenous growth solution

We define a BGP as a path in which consumption, capital, public spending, money, output, public debt, and deficit grow at a common (endogenous) rate ($\gamma^* = \dot{c}/c = \dot{k}/k =$

$\dot{m}/m = \dot{b}/b = \dot{d}/d)$, while the real (r^*) and nominal (R^*) interest rates (and, as a consequence, the inflation rate π^*) are constant. Thus, in the steady-state, the real interest rate is defined by

$$r^* = \frac{(1-\tau)\,\alpha A g_k^{*1-\alpha}}{1+\phi^k Q^*} - \delta, \tag{18}$$

and the rate of economic growth is simply

$$\gamma^* = S\,(r^* - \rho)\,, \tag{19}$$

where, given the standard transversality condition: $\varepsilon\,(\gamma^*) := r^*/\gamma^* = S^{-1} + \rho/\gamma^* > 1$.

In addition, since $d_k^* = \theta y_k^*$, we obtain from (17)

$$(1-\eta)\,\theta A g_k^{*1-\alpha} = \gamma^* b_k^*, \tag{20}$$

and, from the definition of the deficit in Government budget constraint (5)

$$r^* b_k^* = (\theta + \tau)\,A g_k^{*1-\alpha} - g_k^*. \tag{21}$$

3.1. The effect of deficit and monetization in the long-run: some intuition

Proposition 1. *(Deficits and monetization in the long-run) For a given long-run economic growth (γ^*):*

(i) *any increase in the degree of deficit monetization increases the public expenditure to capital ratio in the long-run.*

(ii) *any increase in the deficit target reduces the public expenditure to capital ratio in the long-run if monetization is small (namely $\eta < \bar{\eta}$), but rises it if monetization is large ($\eta > \bar{\eta}$), where: $\bar{\eta} := 1 - 1/\varepsilon\,(\gamma^*) \in (0,1)$.*

Proof. By (19), (20) and (21), the public-spending-to-capital ratio is, in the steady-state

$$g_k^* = \left[(\theta+\tau)\,A - (1-\eta)\,\theta A \varepsilon\,(\gamma^*)\right]^{1/\alpha}, \tag{22}$$

where $\left.\frac{\partial g_k^*}{\partial \eta}\right|_{\gamma^*} > 0$ and, $\left.\frac{\partial g_k^*}{\partial \theta}\right|_{\gamma^*} \geq 0 \Leftrightarrow 1 \geq (1-\eta)\,\varepsilon\,(\gamma^*) \Leftrightarrow \eta \geq \bar{\eta}$. □

From (22) without deficit ($\theta = 0$), we find the solution of Barro (1990), namely: $g_k^* = (\tau A)^{1/\alpha} =: g_k^B$. With deficit but no monetization ($\theta > 0$ and $\eta = 0$), we obtain: $g_k^* = [\tau A - \theta A(\varepsilon(\gamma^*) - 1)]^{1/\alpha} < g_k^B$. Since the standard transversality condition ensures that $\varepsilon\,(\gamma^*) > 1$, for the consumption path to be bounded, the public spending ratio is lower with deficits (and no monetization) than under a balanced budget rule (hereafter BBR), as described by Minea and Villieu (2012). The basic mechanism driving this crowding-out effect is the following. On the one hand, deficits generate a permanent flow of new resources ($\dot{b}$). On the other hand, debt generates a permanent flow of new unproductive expenditures (the debt burden rb). In steady-state, the standard transversality condition ($r^* > \gamma^* = \dot{b}/b$) means that the latter dominates the former ($rb > \dot{b}$), so that any rule that authorizes permanent deficits involves net costs for public finance in the long-run, irrespective of the precise nature of this rule. However, this configuration radically changes if deficit-monetization is authorized. Effectively, in such circumstances, the debt burden can be accommodated by money creation. Suppose, for example, that the deficit is fully monetized ($\eta = 1$ in (22)); compared to the BBR used in Barro (1990), taxes are now supplemented by the deficit: $g_k^* = [(\theta + \tau) A]^{1/\alpha} > g_k^B$ if $\theta > 0$. Intuitively, this is because the new resources provided by the deficit are devoted to productive spending, while the (additional) interest burden is financed by issuing new money.

More generally, Proposition 1 shows that (i) the monetization of fiscal deficits allows to reduce their crowding-out effect on productive expenditures, and (ii) it can even, if large enough, increase the latter. As economic growth positively depends on public expenditures, the impact of deficits on long-run growth is likely to depend on the degree of monetization. However, this result is only preliminary; indeed, in equilibrium $\bar{\eta}$ depends on η (and on other parameters of the model), since γ^* is an endogenous function of parameters, including η. In the following, we present the long-run solution of the model.

3.2. The steady-state

The long-run solution of the model is computed in Appendix C. Endogenous growth solutions are obtained at the intersection of two relations between γ^* and g_k^* . The first relation comes from (19) and (22), namely

$$\gamma^* = \frac{S\rho(1-\eta)\theta A}{S[(\theta+\tau)A - g_k^{*\alpha}] - (1-\eta)\theta A} =: \mathcal{G}(g_k^*). \tag{23}$$

The second relation is simply the Keynes-Ramsey rule (19) where the real interest rate has to be amended by the financing cost of investment, which in turn depends on the nominal interest rate and thus on the components of money equilibrium:

$$\gamma^* = S\left\{[1 - o(\cdot)] \frac{(1-\tau)\,\alpha A g_k^{*1-\alpha}}{1 + \phi^k\,(1+\mu)\,\eta\theta A g_k^{*1-\alpha}/\mu e_k^*} - \rho - \delta\right\}, \tag{24}$$

where $o(\cdot)$ is a negligible term for "small" transaction costs, since $o(\cdot) \to 0$ if $\omega \to 0$. Eq. (24) describes an implicit function between γ^* and g_k

$$g_k^* =: \mathcal{F}(\gamma^*), \tag{25}$$

where $\mathcal{F} \in C^\infty(\mathbb{R}_+^*)$ is an increasing strictly convex function (see Appendix C).

The steady-state solutions can thus be computed as

$$\gamma^* = \mathcal{G}(\mathcal{F}(\gamma^*)). \tag{26}$$

In general, the model exhibits multiplicity: there are two solutions that verify (26). To provide some intuition about this multiplicity, let us first study a special case without deficit or money.

Definition 1. (*Steady-state solutions without public deficit or money*) *Without public deficit or money ($\theta = \omega = 0$) the model gives rise to two solutions: a no-growth solution (that we call the "Solow" solution $\gamma^S = 0$) and a positive growth solution (that we call the "Barro" solution $\gamma^B > 0$).*

We find the "Solow" solution by putting $\theta = 0$ in (23). Consequently, the rate of economic growth is simply $\gamma^S = 0$, and, from the Keynes-Ramsey rule (24), the real interest rate is $r^* = \rho$. The public spending ratio is obtained by (18) (notice that $Q^* = 0$ if $\omega = 0$), namely: $g_k^S = [(\rho + \delta)/\alpha A(1-\tau)]^{1/(1-\alpha)}$. The couple (g_k^S, γ^S) characterizes point S in Figure 1 below. However, there is another long-run solution, which corresponds to Barro (1990), if $g_k^* = g_k^B = (\tau A)^{1/\alpha}$ in (23). This solution gives rise to a "0/0" case of indeterminacy, but the rate of economic growth can easily be computed from (24)

$$\gamma^B = S\left[\alpha A\left(1-\tau\right)\left(A\tau\right)^{(1-\alpha)/\alpha} - \rho - \delta\right]. \tag{27}$$

The Barro solution corresponds to a zero stock of public debt in the steady state $\left(b_k^B = 0\right)$ and is depicted by point B in Figure 1 below.

Intuitively, this multiplicity comes from the fact that Barro (1990) assumes a BBR with zero public debt at any instant (including the initial time $t = 0$), while in our model, the case $\theta = 0$ corresponds to a BBR at any time, but public debt can be positive at date $t = 0$.[11] Thus, if the initial stock of public debt is very high, such that $b_k^S = (g_k^S)^{1-\alpha}[\tau A - (g_k^S)^{\alpha}]/\rho > 0$, economic growth cannot emerge, because most of public resources are diverted from productive use and devoted to the debt burden. It follows that the economy is locked into a poverty trap, namely a no-growth steady-state where public debt remains at its initial level.

On the contrary, if the public debt is initially at zero, the economy can grow at a positive endogenous rate $\gamma^B > 0$, since productive public expenditures are not subject to the crowding-out effect of the debt burden. The question of how the economy converges to these BGPs will be addressed in section 4 below. Let us now turn our attention to the general long-run solutions of the model when $\theta \neq 0$ and $\omega \neq 0$.

Proposition 2. *(Multiplicity of BGPs) For $\theta > 0$ and $0 < \eta < 1$, two and only two BGPs characterize the long-run solution of the model: a high BGP (γ^{*h}) and a low BGP (γ^{*l}), where $0 < \gamma^{*l} < \gamma^{*h}$.*

Proof: See Appendix C.

Relations (23)-(24) are depicted in Figure 1, where point H characterizes the high BGP, while point L denotes the low BGP.

[11] Notice that, with $\theta = 0$, public debt must be constant but not necessarily zero in the long-run. The Solow solution results from the fact that the constant level of public debt forces private capital to be constant in the long-run to achieve a constant steady-state b_k^S ratio. Thus, economic growth disappears in the steady-state $\left(\gamma^S = 0\right)$.

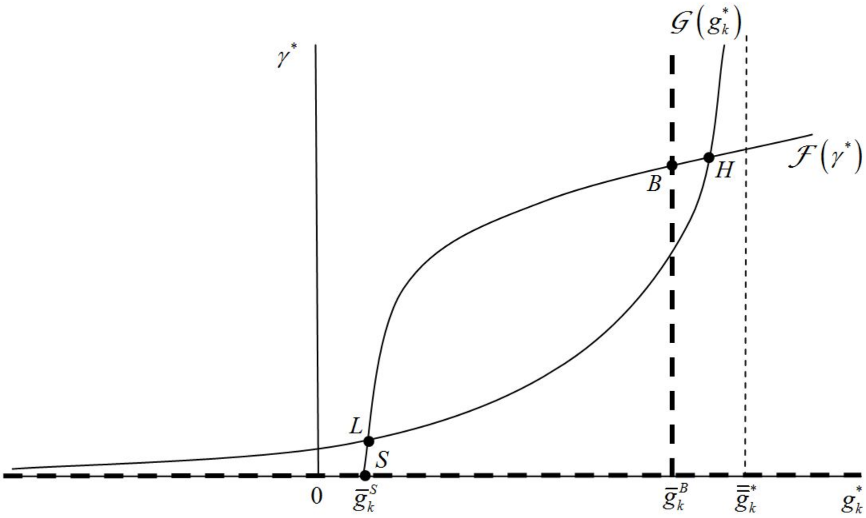


Figure 1: Multiplicity of BGPs

The intuitive explanation of this multiplicity is the following. Economic growth positively depends on the public-expenditures-to-capital ratio, which increases the marginal productivity of private capital in the Keynes-Ramsey rule (24). In addition, public expenditures are an increasing function of economic growth in the Government budget constraint (23), because growth allows to reduce public debt in the long-run (in the steady-state $b_k^* = (1-\eta)\,\theta y_k^*/\gamma^*$). Consequently, the higher the economic growth, the lower the public debt, with an unchanged deficit target (θ). This dual interaction between economic growth and public expenditure generates multiplicity: for the same set of parameters, a high BGP (H) and a low BGP (L) coexist. Effectively, a high growth, by reducing the debt burden, allows increasing public expenditure, which further enhances growth, while low growth magnifies the crowding-out effect of debt on productive public spending, which, in turn, decreases growth.

3.3. Deficits and monetization in the long-run

Proposition 3. *(The effect of deficit in the steady-state)*

(i) *Along the low BGP (γ^{*l}), any upwards shift in the deficit target (θ) reduces economic growth;*

(ii) *Along the the high BGP (γ^{*h}), there is a critical level of the degree of monetization η (say, $\bar{\eta}^h$) such that any upwards shift in the deficit target (θ) reduces economic growth if monetization is small ($\eta < \bar{\eta}^h$), but increases it if monetization is large ($\eta > \bar{\eta}^h$).*

Proof. See Appendix D.

From (26), we can define the following implicit function

$$\mathcal{H}(\gamma^*) := \mathcal{G}(\mathcal{F}(\gamma^*)) - \gamma^* \;\; = 0. \tag{28}$$

Using the implicit function Theorem, the effect of the deficit ratio on the BGPs can be obtained as

$$\left.\frac{d\gamma^*}{d\theta}\right|_{\gamma^*=\gamma^{*i}} = -\left.\frac{\partial_\theta \mathcal{H}(\gamma^*,\theta)}{\partial_\gamma \mathcal{H}(\gamma^*,\theta)}\right|_{\gamma^*=\gamma^{*i}},\ i=\{h,l\},$$

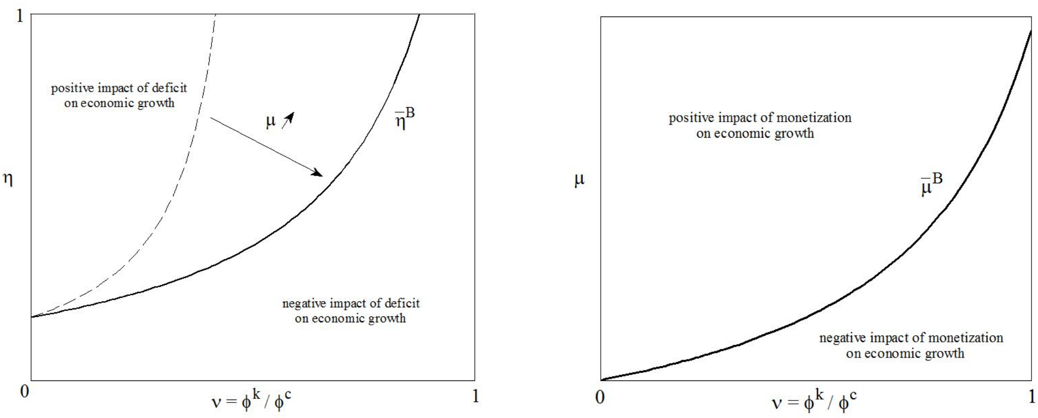
where, for $(\theta,\omega)\to(0,0)$

$$\begin{cases} \partial_\gamma \mathcal{H}(\gamma^*,\theta)|_{\gamma^*=\gamma^{*l}} \to \partial_\gamma \mathcal{H}(\gamma^*,\theta)|_{\gamma^*=\gamma^S} > 0 \\ \partial_\gamma \mathcal{H}(\gamma^*,\theta)|_{\gamma^*=\gamma^{*h}} \to \partial_\gamma \mathcal{H}(\gamma^*,\theta)|_{\gamma^*=\gamma^B} < 0 \end{cases}. \tag{29}$$

In addition, we have $\text{Sign}\{d\gamma^*/d\theta|_{\gamma^*=\gamma^{*i}}\} = \text{Sign}\{\eta-\bar{\eta}^i\},\ i=\{h,l\}$, where, defining $v := \phi_k/\phi_c$ and $x\left(\gamma^i\right) := 1-\tau-(1-v)\left(\gamma^i+\delta\right)A^{\frac{-1}{\alpha}}\tau^{\frac{\alpha-1}{\alpha}}$:

$$\bar{\eta}^i = \frac{\left(\frac{1-\alpha}{\alpha}\right)x\left(\gamma^i\right)\left[\varepsilon\left(\gamma^i\right)-1\right]}{\left(\frac{1-\alpha}{\alpha}\right)x\left(\gamma^i\right)\varepsilon\left(\gamma^i\right)-\tau v\left(\frac{1+\mu}{\mu}\right)}. \tag{30}$$

In the neighborhood of the Solow BGP ($\gamma^S=0$), $\sigma(\gamma^S)\to+\infty$ and $\bar{\eta}^S\to 1$, so that, since $\partial_\gamma\mathcal{H}(\gamma^*,\theta)|_{\gamma^*=\gamma^S}>0$ in (29): $\text{Sign}\{\frac{d\gamma^*}{d\theta}|_{\gamma^*=\gamma^S}\} = \text{Sign}\{1-\eta\}\geq 0$, for any $\eta\leq 1$, which proves point (i). In the neighborhood of the Barro BGP, $\bar{\eta}^h\to\bar{\eta}^B$, with $\partial_v\tilde{\eta}^B\geq 0$, $\partial_\mu\tilde{\eta}^B\leq 0$, and where $\bar{\eta}^B$ is defined in (30) for $i=B$, and, since $\partial_\gamma H(\gamma^*,\theta)|_{\gamma^*=\gamma^B}<0$ in (29), $\text{Sign}\{\frac{d\gamma^*}{d\theta}|_{\gamma^*=\gamma^B}\} = \text{Sign}\{\eta-\bar{\eta}^B\}$, which proves point (ii).[12] □



-a- Increase in the deficit ratio -b- Increase in monetization

Figure 2: Comparative statics along the high BGP

Along the high BGP, an increase in the deficit ratio is all the more likely to have a positive effect on economic growth that money demand is inelastic to interest rate and

[12]Relation (30) provides an explicit value for $\bar{\eta}^B$ because γ^B is independent of η.

the transaction costs on investment are not too high. Effectively, any increase in the interest-elasticity of the demand for money $(1/\mu)$ or in transaction costs on investment (υ) moves up the border $\bar{\eta}^B$ and reduces the parameters space for which the deficit has a positive impact (see Figure 2a). These results are very intuitive. The deficit has a positive effect on growth because it allows for more productive public spending, but a negative effect because it increases the burden of the public debt. This negative effect can be mitigated by monetizing deficits, but this also entails costs, because it increases the nominal interest rate and transaction costs in the economy. If private investment is highly subject to transaction costs or money demand responds strongly to the interest rate, the gain of monetizing deficits will be very low.

Let us now turn our attention to the direct effect of monetization on the BGPs.

Proposition 4. *(The effect of monetization in the steady-state)*

(i) *Along the low BGP (γ^{*l}), any upwards shift in the rate of deficit monetization reduces economic growth;*

(ii) *Along the high BGP (γ^{*h}), there is a critical level of the interest-elasticity of money demand μ (say, $\bar{\mu}^h$) such that any upwards shift in the rate of deficit monetization increases economic growth if the interest-elasticity of money demand is small $\left(\mu > \bar{\mu}^h\right)$, but decreases it this elasticity is large $\left(\mu \ < \bar{\mu}^h\right)$.*

Proof. See Appendix D.
Using the implicit function Theorem in (28), the effect of monetization on the BGPs can be obtained as

$$\left.\frac{d\gamma^*}{d\eta}\right|_{\gamma^*=\gamma^{*i}} = -\left.\frac{\partial_\eta \mathcal{H}(\gamma^*,\eta)}{\partial_\gamma \mathcal{H}(\gamma^*,\eta)}\right|_{\gamma^*=\gamma^{*i}}, \ i=\{h,l\},$$

where, for $(\theta,\omega)\to(0,0)$, $\text{Sign}\{\frac{d\gamma^*}{d\eta}|_{\gamma^*=\gamma^{*i}}\} = \text{Sign}\{\mu-\bar{\mu}^i\}, i=\{h,l\}$ with,

$$\bar{\mu}^i = \frac{\tau\upsilon}{\left(\frac{1-\alpha}{\alpha}\right)x(\gamma^{*i})\varepsilon(\gamma^{*i})-\tau\upsilon}. \tag{31}$$

In the neighborhood of the Solow BGP $(\gamma^S=0)$, $\varepsilon(\gamma^S)\to+\infty$ and $\bar{\mu}^S\to 0$, so that $\partial_\eta\mathcal{H}(\gamma^*,\eta)_{\gamma^*=\gamma^S}<0$ and, by (29), $\frac{d\gamma^*}{d\eta}|_{\gamma^*=\gamma^S}<0$, for $\mu>0$, which proves point (i). In the neighborhood of the Barro BGP, $\bar{\mu}^h\to\bar{\mu}^B$, with $\partial_\upsilon\tilde{\mu}^B\geq 0$, where $\bar{\mu}^B$ is defined in (31) for $i=B$, and $\text{Sign}\{\frac{d\gamma^*}{d\eta}|_{\gamma^*=\gamma^B}\}=\text{Sign}\{\mu-\bar{\mu}^B\}$, which proves point (ii).[13] □

Along the high BGP, monetization has a favorable effect on economic growth if the demand for money is not very elastic to the interest rate. This is notably the case in the CIA special case where $\mu\to+\infty$. If the money demand is very elastic to the interest rate $\left(\mu<\bar{\mu}^B\right)$, however, monetization impedes economic growth, because it creates inflation and increases transaction costs in the long-run. This effect is related to the importance of transaction costs on investment (υ). If investment is not subject to transaction costs $(\upsilon=0)$, the effect of monetization on the high BGP is always positive $\left(\mu^B=0\right)$, because there is no more trade-off between the financing costs and the debt burden reduction of monetization. But the trade-off reappears as soon as $\upsilon>0$, as shows Figure 2b.

[13] Relation (31) provides an explicit value for $\bar{\mu}^B$ because γ^B is independent of μ.

One interesting question is why only the high BGP is subject to threshold effects in line with the level of deficit or monetization. To elucidate this question, we have to return to equation (23) above. Clearly, in this relation, the effect of deficit or monetization on economic growth can reverse only if the denominator changes sign, i.e. if $g_k^* > \bar{g}_k := [(\theta+\tau)A - (1-\eta)\theta A/S]^{1/\alpha}$. In the neighborhood of the low BGP, the public spending ratio is so low that $g_k^{*l} < \bar{g}_k$. Consequently, the impact of changes in the deficit ratio or in the degree of monetization follows the direct effect in the numerator of (23), namely: $d\gamma^*/d\theta > 0$ and $d\gamma^*/d\mu < 0$.

4. Transitional dynamics and indeterminacy

Outside the steady-state, the model gives rise to a five-variable reduced form, which can be solved recursively (see Appendix B), namely, for $\phi^k \zeta(g_k) \neq 0$:

$$\begin{cases} (a) & \dot{d}_y = -\xi\left(d_y - \theta\right) \\ (b) & \dot{b}_k = (1-\eta)\, d_y A g_k^{1-\alpha} - \gamma_k b_k \\ (c) & \dot{Q} = \frac{1}{\phi^k}\left[(r+\delta)\left(1+\phi^k Q\right) - (1-\tau)\,\alpha A g_k^{1-\alpha}\right] \\ (d) & \dot{c}_k = S\left[r - \rho - \phi^c \dot{Q}/(1+\phi^c Q)\right] c_k - \gamma_k c_k \\ (e) & \dot{g}_k = \frac{1}{\phi^k \zeta(g_k)}\left\{\eta d_y A g_k^{1-\alpha}\left(\frac{Q}{\omega}\right)^{\frac{1}{\mu}} + \left(r - R - \gamma_k + \frac{\dot{Q}}{\mu Q}\right) e_k - \left(\phi^c - \phi^k\right)\dot{c}_k\right\} \end{cases} \tag{32}$$

where $R = \frac{\omega\mu}{1+\mu}\left(\frac{Q}{\omega}\right)^{(1+\mu)/\mu}$, and

$$\gamma_k = A g_k^{1-\alpha} - g_k - c_k - \delta =: \gamma(c_k, g_k), \tag{33}$$

$$e_k = \left(\phi^c - \phi^k\right) c_k + \phi^k\left(A g_k^{1-\alpha} - g_k\right) =: e\left(c_k, g_k\right), \tag{34}$$

$$r = \left(\left(d_y + \tau\right) A g_k^{1-\alpha} - g_k\right)/b_k =: r\left(d_y, g_k, b_k\right), \tag{35}$$

$$\zeta\left(g_k\right) := \frac{d}{dg_k}\left(A g_k^{1-\alpha} - g_k\right) = (1-\alpha)\, A g_k^{-\alpha} - 1. \tag{36}$$

As the public debt ratio b_k and the deficit to output ratio d_y cannot jump, there are 2 predetermined variables in this system.[14]

4.1. A special case: transaction costs on consumption only

In the special case where the transaction technology does not affect investment goods $\left(\phi^k = 0\right)$, relations (32c) and (32d) are not defined, and the reduced form becomes a

[14]Effectively, the deficit-to-output ratio (d_y) cannot jump at any time, because it is defined by the smooth adjustment dynamics (6). Moreover, the debt-to-output ratio $b_k = b/k$ cannot jump, because the stocks of public debt (b) and capital (k) are predetermined at each instant.

four-variable one, namely (see Appendix B)

$$\left\{\begin{array}{ll} (a) & \dot{d}_y = -\xi\left(d_y - \theta\right) \\ (b) & \dot{b}_k = \left(1-\eta\right) d_y A g_k^{1-\alpha} - \gamma_k b_k \\ (c) & \dot{Q} = \frac{\mu Q(1+\phi^c Q)}{1+(1+\mu S)\phi^c Q}\left[\frac{\omega\mu}{1+\mu}\left(\frac{Q}{\omega}\right)^{\frac{1+\mu}{\mu}} - \frac{\eta d_y A g_k^{1-\alpha}}{\phi^c c_k}\left(\frac{Q}{\omega}\right)^{\frac{1}{\mu}} - \left(1-S\right) r - \rho S\right] \\ (d) & \dot{c}_k = S\left[r - \rho - \phi^c \dot{Q}/(1+\phi^c Q)\right] c_k - \gamma_k c_k \end{array}\right. \tag{37}$$

The crucial difference relative to (32) is that the public spending ratio is no longer part of the reduced form, but is obtained by the means of (35), which rewrites

$$g_k := g\left(d_y, b_k\right), \tag{38}$$

while the real interest rate is simply (from (32c))

$$r = \left(1-\tau\right) \alpha A g_k^{1-\alpha} - \delta =: r\left(g_k\right). \tag{39}$$

The linearization of (37) in the neighborhood of BGPs provides the following system

$$\left(\begin{array}{c} \dot{d}_y \\ \dot{b}_k \\ \dot{Q} \\ \dot{c}_k \end{array}\right) = \mathbf{J}_1^i \left(\begin{array}{c} d_y - d_y^{*i} \\ b_k - b_k^{*i} \\ Q - Q^{*i} \\ c_k - c_k^{*i} \end{array}\right), \quad i = \left\{h, l\right\}, \tag{40}$$

where $\mathbf{J}_1^i$ stands for the Jacobian matrix in the neighborhood of BGP $i = h, l$. According to Blanchard-Kahn conditions, the steady-state is (saddle-path) stable and well determined if $\mathbf{J}_1^i$ contains exactly 2 negative eigenvalues (with one eigenvalue equal to $-\xi$) and 2 positive eigenvalues.

Proposition 5. *(Stability) For small values of the deficit ratio (formally $\theta \to 0^+$)*

(i) *the low BGP is unstable and,*
(ii) *the high BGP is saddle-path stable.*

Proof. See Appendix E.
Appendix E shows that in the neighborhood of the Barro BGP $\mathbf{J}_1^S$ contains two positive and two negative eigenvalues, while in the neighborhood of the Solow BGP $\mathbf{J}_1^S$ contains one negative and three positive eigenvalues. By continuity, these properties are verified for positive (and low) values of the deficit ratio. Therefore, the high BGP is well determined, while the low BGP is unstable. □

Consequently, we can exclude multiplicity on the basis of the analysis of local dynamics: the high BGP is the only relevant equilibrium path when investment is not subject to transaction costs. This is no longer the case in the general version of the model, as we will see.

4.2. Transaction costs on consumption and investment

The linearization of (32) in the neighborhood of BGPs provides the following system

$$\left(\begin{array}{c} \dot{d}_y \\ \dot{b}_k \\ \dot{Q} \\ \dot{c}_k \\ \dot{g}_k \end{array}\right) = \mathbf{J}_2^i \left(\begin{array}{c} d_y - d_y^{*i} \\ b_k - b_k^{*i} \\ Q - Q^{*i} \\ c_k - c_k^{*i} \\ g_k - g_k^{*i} \end{array}\right), \quad i = \{h, l\}, \tag{41}$$

where $\mathbf{J}_2^i$ stands for the Jacobian matrix in the neighborhood of BGP $i = h, l$. According to Blanchard-Kahn conditions, $\mathbf{J}_2^i$ must contain 2 negative eigenvalues (with one eigenvalue equal to $-\xi$) and 3 positive eigenvalues.

Proposition 6. *(Multiplicity and indeterminacy) For small values of the deficit ratio (formally $\theta \to 0^+$)*

(i) *the low BGP is saddle-path stable and,*
(ii) *the high BGP is locally undetermined or saddle-path, depending on parameters.*

Proof. See Appendix F.
The inspection of the reduced form (32) reveals that the dynamics fundamentally shift with the value of the term $\zeta(g_k^*)$. Effectively, the system is not defined for $\zeta(g_k^*) = 0$, and the determinant of the Jacobian matrix $\mathbf{J}_2^i$ changes sign whenever $\zeta\left(g_k^{*i}\right)$ changes sign. Yet, to be fully determined, the BGP must be associated to exactly 2 negative eigenvalues. As the determinant of the Jacobian matrix is the product of the 5 eigenvalues, such a configuration is possible only if it is positive, i.e. BGP determinacy cannot be ensured when the determinant is negative. On the one hand, Appendix F shows that, in the neighborhood of the Solow BGP (i.e. $g_k \to g_k^S$), $\zeta(g_k) > 0$, for any g_k and $\mathbf{J}_2^S$ has exactly 2 negative and 3 positive eigenvalues; thus the Solow BGP is saddle path. By continuity, this must be true for positive (but small) values of the deficit ratio. This proves point (i). On the other hand, in the neighborhood of the Barro BGP (i.e. $g_k \to g_k^B$), on the contrary, $\zeta(g_k)$ changes sign depending on parameters and $\mathbf{J}_2^B$ has 2 negative and 3 positive eigenvalues if $\zeta(g_k) < 0$, but 3 negative and 2 positive eigenvalues if $\zeta(g_k) > 0$. By continuity, this property is verified for positive (but low) values of the deficit ratio. In the former case the high BGP is well determined, while in the latter it exhibits local indeterminacy. This proves point (ii). □

In the neighborhood of the low BGP, the public spending ratio is very small (see Figure 1), hence $\zeta\left(g_k^l\right) > 0$, for any g_k^l and the BGP is well determined. This is not true in the neighborhood of the high BGP. The high BGP is locally determined only if $\zeta\left(g_k^{*h}\right) < 0$, namely if $g_k^{*h} > \hat{g}_k^* := A^{1/\alpha}(1-\alpha)^{1/\alpha}$. By (22), this criterion amounts to

$$\tau > 1 - \alpha - \theta\left[1 - (1-\eta)\,\varepsilon\left(\gamma^h\right)\right]. \tag{42}$$

If $\theta = 0$, to ensure the determinacy of the high BGP, the tax-rate must be higher than the Barro (1990) optimal rate, namely $\tau^B := 1 - \alpha$. This is no longer the case with public deficit, since determinacy can be ensured with tax-rates lower than τ^B, especially if monetization is high. The higher the monetization degree, the more likely the equilibrium is determined, for given tax rates and deficit ratios. Figure 3 synthesizes these results.[15] Determinacy is all the more likely to occur (i) the higher the monetization degree, and (ii) the higher the tax rate (since θ is small in (42)). Determinacy of the high BGP is ensured above the AA line, which pivots downward as deficit monetization increases.

Proposition 6 shows that multiplicity cannot be removed on the basis of local dynamics of the two BGPs. On the contrary, if transactions costs affect both consumption and investment expenditures, both steady-states are reachable, depending on the initial level of the public debt ratio. If the initial public debt ratio is "high", the economy will be attracted by the low-growth BGP, and will continue its trajectory in the near context of a poverty trap with economic growth approaching zero. If the initial public debt ratio is "low", on the contrary, the economy will converge towards the high BGP, but the exact path that the economy will follow during the transition may be subject to sunspot equilibria, i.e. the existence of a continuum of equilibrium paths converging towards the high BGP, starting from the same initial value of state variables. In such a case, as suggests Figure 3, deficit monetization can be used as a selection device to solve indeterminacy and to obtain a unique transition path. This analysis shows that studying indeterminacy in a monetary growth model can provide useful insights to explore the role of monetary policy as a tool for macroeconomic stabilization, following the intuition of Suen and Yip (2005).

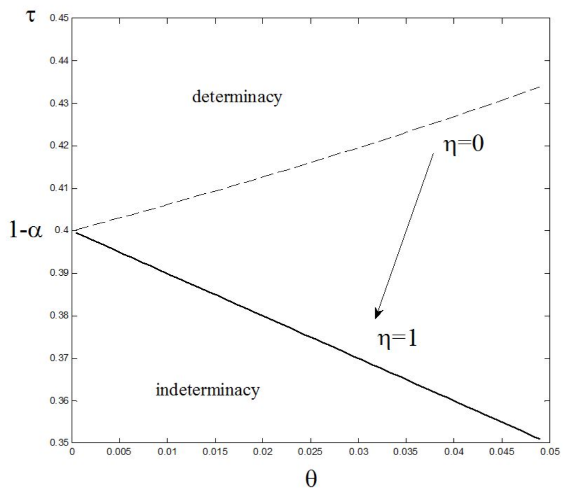


Figure 3: Determinacy and indeterminacy of the high BGP

[15] In order to illustrate our analytically-established results, Figure 3 is built on usual numerical values $S = A = \phi_c = \phi_k = \mu = 1$, $\alpha = 0.6$, $\rho = \delta = \omega = 0.05$, showing that indeterminacy arises for non-exotic parameter values. Of course, Proposition 6 and equation (42) are independent of simulation values.

4.3. Discussion

Our general result is that the high BGP is in some sense "more stable" than the low BGP. Basically, this property is due to the behavior of the public debt ratio in the Government budget constraint. Effectively, the dynamics of the public debt ratio are driven by the difference between the debt burden and economic growth. As usual in the analysis of Government budget constraint, a sufficiently high economic growth rate allows to circumvent the inherent unstable dynamics of public debt, thus stabilizing the public debt ratio. This is the case in the neighborhood of the high BGP. On the contrary, along the low BGP, economic growth is very low and cannot stabilize the evolution of the public debt ratio. This explains why, in the special case with transaction costs on consumption only, the high BGP is saddle-point stable, while the low BGP is unstable.

In the general version of the model, the same reasoning applies, but the reduced form of the model has one additional equation, based on the evolution of a jumpable variable (the public spending ratio g_k), which can create indeterminacy. Indeed, in this reduced form (system (32)), the nature of the dynamics of $\dot{g}_k$ fundamentally shifts with respect to $\zeta(g_k)$. The intuitive explanation of this shift is the following. The term $\zeta(g_k) = d(y_k - g_k)/dg_k$ is the response of the difference between output and public spending following an increase in public spending, or, in other words, the net impact of an additional unit of productive public expenditure on the goods market equilibrium. Thus, any rise in g_k increases (decreases) private demand if $\zeta(g_k) > 0$ ($\zeta(g_k) < 0$). Since money demand comes from private expenditures (consumption plus investment, e_k), money demand positively (if $\zeta(g_k) > 0$) or negatively (if $\zeta(g_k) < 0$) depends on productive public expenditure.

Yet, in the money market equilibrium, money emissions are defined by the difference between the monetization of public deficit and the inflation tax ($\dot{m}_k = \eta d_k - (\pi + \gamma)m_k$). Suppose an upward jump in public spending from the high BGP, such that $g_k > g_k^*$, *ceteris paribus*. As a result of the excess demand in goods equilibrium, the inflation rate jumps up, and seignorage collection ($(\pi + \gamma)m_k$)) becomes higher than deficit monetization (ηd_k). Thus, the emission of money becomes negative ($\eta d_k < (\pi + \gamma)m_k \Rightarrow \dot{m}_k < 0$). In equilibrium, money demand must decline, thus private demand must decrease ($\dot{e}_k < 0$), which implies: $\dot{g}_k < 0$ if $\zeta(g_k) > 0$, or $\dot{g}_k > 0$ if $\zeta(g_k) < 0$.

In the first case, the law of motion of g_k is stable, leading to indeterminacy of the BGP (recall that g_k is a jumpable variable), while in the latter, the law of motion of g_k is unstable, leading to determinacy of the BGP. Hence, for configurations of parameters satisfying (42), the public spending ratio becomes so large that the derivative $\zeta(g_k)$ becomes negative. As a result of this novel source of instability, the high BGP loses its undesirable property of being stable and undetermined, and becomes saddle-path stable.

Observe that our indeterminacy result it quite general, compared to the literature. In one sector "Ak"-type endogenous growth models, Suen and Yip (2005) and Chen and Guo (2008) show that local indeterminacy is due to the presence of an intertemporal substitution effect on capital accumulation, whose strength positively depends on the intertemporal elasticity of substitution in consumption, while (Jha et al., 2002) find that the technology is a key determinant of the stability of the equilibrium. In two-sector endogenous growth models, using a discrete-time approach, Bosi et al. (2010) show that indeterminacy crucially depends on the timing of (intra-period) monetary arrangements and on the specification of preferences. In contrast, our indeterminacy result is not sensitive to the consumption elasticity of substitution, nor to the form of the utility function. It is not more sensitive to the timing of monetary payments, since in continuous time, any intra-period mechanism disappears. Moreover, in our setting, indeterminacy does not depend on the interest-elasticity of money demand: indeterminacy arises if investment goods are subject to transaction costs, but not if consumption only is affected, independently of the interest-elasticity of money demand (and in particular in the CIA special case with a zero-elasticity). This feature outlines the interest to introduce a general transaction cost technology that includes capital goods in economic growth models, as pioneered by (Palivos and Yip, 1995). Finally, compared to Bosi and Magris (2003) and Bosi et al. (2005), who show the importance of having a partial CIA constraint on consumption goods (namely $\phi^c < 1$ or $\phi^c = 1$), or Chen and Guo (2008) and Bosi and Dufourt (2008), who point out the role of a fractional CIA constraint on investment, our indeterminacy result does not depend on the exact fraction of investment expenditures that are subject to transaction costs (provided it is strictly positive: $\phi^k > 0$).

Thus, our analysis, which focuses on the interaction between deficit monetization and the form of the money demand (and especially the way it reacts to changes in public expenditures in goods market equilibrium), goes beyond existing studies, emphasising increasing returns in the production function, the timing or the fraction of transactions that are subject to cash requirement, or the value of the intertemporal elasticity of substitution in the utility function as a source of multiplicity and indeterminacy.

5. Conclusion

Introducing public debt and deficit monetization in an endogenous growth model with productive public expenditure can lead to multiplicity and indeterminacy. In our model, multiplicity refers to the coexistence of two achievable BGPs in the long-run: a high BGP and a low BGP. Indeterminacy refers to the transition path towards the high BGP, which is locally indeterminate for a large range of parameters.

Overall, from an economic policy standpoint, our results provide two new motivations for monetizing deficits. On the one hand, along the high BGP, monetization can be useful because it avoids (or limits) the crowding-out effect of public indebtedness on productive public expenditure in the long-run. Usually, monetization is defended for providing seigniorage revenues, or because inflation surprises can reduce the cost of capital. Yet, seigniorage revenues are fairly small and inflation surprises cannot be perpetuated in rational expectation equilibria, thus our motivation for monetizing deficits to increase public spending might be stronger. Indeed, in our model, money issuance increases economic growth on the long-run perfect-foresight BGP, owing to a composition effect in public finance, namely the substitution of a non-interest-bearing asset (money) to public debt in Government budget constraint. This change in the composition of Government liabilities generates a less distortive way of finance for productive public expenditure. However, monetization also produces distortions, by increasing transaction costs, and its positive effect on growth only holds if the interest-elasticity of money demand is sufficiently low. Therefore, monetization can be viewed as a monetary policy support, rather than the ultimate tool for promoting long-run economic growth.

On the other hand, with transaction costs on investment, transitional dynamics in the neighborhood of the high BGP crucially depend on the rate of monetization. For small monetization rates, the low BGP is locally determined (saddle-path), but the high BGP becomes locally undetermined. However, for sufficiently high monetization rates, both BGPs are characterized by the saddle-path property and are locally determined. Thus, a large dose of monetization might allow avoiding, whenever present, BGP indeterminacy. Our findings match numerous results in the literature, emphasizing the importance of the transaction technology in generating long-run multiplicity and/or indeterminacy of perfect-foresight equilibria, and provide an original mechanism according to which deficit monetization could be used as a selection device to solve indeterminacy and to obtain a unique transition path.

Of course, the issue of deficit monetization deserves future research. One strand of work could explore the way endogenous taxes, in addition to monetization, impact the deficit-growth relationship, and act as a potential second source of multiplicity or indeterminacy. Moreover, one could take a closer look at the type of public spending financed by deficit monetization, e.g. by considering public capital or differentiating between productive and unproductive public expenditure, possibly in a two-sector model. Finally, our setup provides an appropriate environment for studying the dynamic strategic interaction between monetary and fiscal policies in a context of growing public debt.

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Appendix A. Solution of Households program.

The representative Household maximizes (1) subject to (2)-(3)-(4), k_0 and b_0 given, and the transversality condition: $\lim_{t\to\infty}\{\exp(-\int_0^\infty r_s\, ds)\,(k_t+b_t+m_t)\}=0$. Since investment is subject to transaction costs, it is convenient to replace the budget constraint (4) by two constraints on two state variables: $a_t := m_t + b_t$ and k_t, using the definition of net investment: $\dot{k}_t = z_t - \delta k_t$. Thus, we can write the current Hamiltonian as

$$\begin{aligned} H_c = u\left(c_t\right) + \lambda_{1t}\left[r_t b_t + \left(1-\tau\right) y_t - c_t - z_t - T\left(c_t, z_t, m_t\right) - \pi_t m_t + l_t\right] \\ +\lambda_{2t}\left(z_t - \delta k_t\right) + \lambda_{3t}\left(a_t - b_t - m_t\right), \end{aligned}$$

where λ_{1t} and λ_{2t} are the co-state variables associated with a_t and k_t, respectively, and λ_{3t} is the Lagrange multiplier associated with the static constraint. The first-order conditions (hereafter FOC) are

$$
\begin{aligned}
&/b_t && \lambda_{3t}/\lambda_{1t} = r_t, && \text{(A.1)}\\
&/c_t && u_c(c_t) = \lambda_{1t}(1+T_c(\cdot)) = \lambda_{1t}(1+\phi^c Q_t), && \text{(A.2)}\\
&/z_t && \lambda_{2t} = \lambda_{1t}(1+T_z(\cdot)) = \lambda_{1t}\left(1+\phi^k Q_t\right), && \text{(A.3)}\\
&/m_t && T_m(\cdot) - \pi_t - \lambda_{3t}/\lambda_{1t} = 0 \Rightarrow \frac{\omega\mu}{1+\mu}\left(\frac{e_t}{m_t}\right)^{1+\mu} = r_t + \pi_t = R_t, && \text{(A.4)}\\
&/a_t && \dot{\lambda}_{1t}/\lambda_{1t} = \rho - r_t, && \text{(A.5)}\\
&/k_t && \dot{\lambda}_{2t}/\lambda_{2t} = \rho + \delta - \frac{(1-\tau) f_k \lambda_{1t}}{\lambda_{2t}} = \rho + \delta - \frac{(1-\tau) f_k}{1+\phi^k Q_t}, && \text{(A.6)}
\end{aligned}
$$

where $Q_t := \omega\,(e_t/m_t)^{\mu}$, $R_t = (\mu/(1+\mu))Q_t^{(1+\mu)/\mu}\omega^{-1/\mu}$ and $e_t := \phi^c c_t + \phi^k z_t$.

These FOCs have a standard interpretation. λ_1 is the shadow price (i.e. the opportunity cost) of financial wealth (a_t), which differs from the shadow price of capital (λ_2) in (A.3), if investment expenditures are subject to transaction costs (namely if $\phi^k > 0$). Effectively, in this case, wealth cannot directly buy capital, because the latter must be acquired with money: the opportunity cost of capital is higher than the opportunity cost of wealth, as soon as $\phi^k > 0$. If capital is not subject to transaction costs, this feature disappears, and $\lambda_{1t} = \lambda_{2t}$. Similarly, in (A.2), the marginal utility of consumption has to be distinguished from the shadow price of financial wealth, since wealth cannot directly buy consumption goods. The opportunity cost of money for consumption ($\phi^c Q_t$) introduces a wedge between the marginal utility of consumption and the marginal value of wealth. Equation (A.4) states that the marginal cost of money (the nominal interest rate R_t) must equalize its marginal return (the marginal value of a unit of money in the transaction costs function. Finally, equations (A.5) and (A.6) describe the evolution of the shadow prices of wealth and capital, respectively. They show, in particular, that the marginal return of capital (its marginal productivity net from taxes and depreciation $(1-\tau) f_k(\cdot) - \delta$) differs from the marginal return of bonds (the real interest rate r_t), as soon as transaction costs affect capital goods.

By differentiating (A.2) and (A.3) and after some simple manipulations we obtain equations (9) and (10) of the main text.

Appendix B. The reduced form of the model.

The definition of the deficit rule (6) provides the first equation of the reduced form

$$\dot{d}_y = -\xi\,(d_y - \theta)\,. \tag{B.1}$$

From (8), we obtain the second equation

$$\dot{b}_k = (1-\eta)\, y_k d_y - \gamma_k b_k. \tag{B.2}$$

With $\phi^k > 0$, the behavior of the nominal interest rate factor directly results from (10)

$$\dot{Q} = \frac{1}{\phi^k}\left[(r+\delta)\left(1+\phi^k Q\right) - (1-\tau)\,\alpha A g_k^{1-\alpha}\right], \tag{B.3}$$

which constitutes the third equation of the reduced form (32). From the Keynes-Ramsey rule (9), we have

$$\frac{\dot{c}_k}{c_k} = S\left[r - \rho - \frac{\phi^c \dot{Q}}{1 + \phi^c Q}\right] - \gamma_k, \tag{B.4}$$

which is the fourth equation of the reduced-form. Finally, we just have to find the path of the public spending ratio g_k. To this end, we compute the time-derivative of (14): $\dot{e}_k = \phi^k \zeta(g_k)\dot{g}_k + \left(\phi^c - \phi^k\right)\dot{c}_k$, with $\zeta(g_k) := (1-\alpha)Ag_k^{-\alpha} - 1$. In money equilibrium, the laws of motion of money supply and demand must coincide, namely, from (13) and (16):

$$\frac{\dot{m}_k}{m_k} = \eta \frac{y_k d_y}{m_k} + r - R - \gamma_k = \frac{\dot{e}_k}{e_k} - \frac{1}{\mu}\frac{\dot{Q}}{Q}, \tag{B.5}$$

it follows that, for $\phi^k \zeta(g_k) \neq 0$:

$$\dot{g}_k = \frac{\dot{e}_k - \left(\phi^c - \phi^k\right)\dot{c}_k}{\phi^k \zeta(g_k)}, \tag{B.6}$$

which corresponds to the last equation of the reduced form (32).

In the case with $\phi^k = 0$, relations (B.3) and (B.6) are not defined. From (10) we find the definition of the real interest rate

$$r = (1-\tau)\alpha A g_k^{1-\alpha} - \delta =: r(g_k), \tag{B.7}$$

which substitutes to (35), and the public spending ratio comes from the definition of the deficit in Government budget constraint (5): $r(g_k)b_k = (d_y + \tau)Ag_k^{1-\alpha} - g_k$, which gives rise to an implicit definition of g_k, namely $g_k := g(d_y, b_k)$. The money equilibrium (B.5) is unchanged, with $e_k = \phi^c c_k$, hence the reduced form (37).

Appendix C. The steady-state solution.

We find the steady-state solution by imposing $\dot{d}_y = \dot{b}_k = \dot{Q} = \dot{c}_k = \dot{g}_k = 0$ in (B.1-B.6). The long-run deficit ratio is $d_y^* = \theta$ and the long-un debt to capital ratio is $b_k^* = (1-\eta)y_k^* d_y^*/\gamma^*$. In steady-state, the nominal interest rate factor is defined as

$$Q^* = \frac{1}{\phi^k}\left[\frac{(1-\tau)\alpha A g_k^{*1-\alpha}}{\frac{\gamma^*}{S} + \rho + \delta} - 1\right], \tag{C.1}$$

and, from (B.2) and (B.4) we obtain the long-run public spending ratio

$$g_k^* = \left[A(\theta + \tau) - A(1-\eta)\theta\left(\frac{1}{S} + \frac{\rho}{\gamma^*}\right)\right]^{1/\alpha}. \tag{C.2}$$

Finally, by (B.5), we obtain

$$Q^* = \frac{1+\mu}{\mu}\left[\frac{\eta\theta A g_k^{*1-\alpha}}{e_k^*} + \left(\rho + \left(\frac{1-S}{S}\right)\gamma^*\right)\left(\frac{Q^*}{\omega}\right)^{-\frac{1}{\mu}}\right], \tag{C.3}$$

and moreover

$$\left(1+\phi^k Q^*\right) = 1+\phi^k\left(\frac{1+\mu}{\mu}\right)\frac{\eta\theta A g_k^{*\,1-\alpha}}{e_k^*} + \phi^k\left(\frac{1+\mu}{\mu}\right)\left(\rho+\left(\frac{1-S}{S}\right)\gamma^*\right)\left(\frac{Q^*}{\omega}\right)^{-\frac{1}{\mu}}.$$

Reintroducing this relation in (C.1), we find

$$\frac{\gamma^*}{S}+\rho+\delta = \frac{(1-\tau)\,\alpha A g_k^{*\,1-\alpha}}{1+\phi^k\left(\frac{1+\mu}{\mu}\right)\frac{\eta\theta A g_k^{*\,1-\alpha}}{e_k^*}+\phi^k\left(\frac{1+\mu}{\mu}\right)\left(\rho+\left(\frac{1-S}{S}\right)\gamma^*\right)\left(\frac{Q^*}{\omega}\right)^{-\frac{1}{\mu}}}, \tag{C.4}$$

and we obtain (24) by rearranging (C.4)

$$\gamma^* = S\left\{\left[1-o\left(\gamma^*,g_k^*\right)\right]\frac{(1-\tau)\,\alpha A g_k^{*\,1-\alpha}}{\left[1+\phi^k\left(\frac{1+\mu}{\mu}\right)\frac{\eta\theta A g_k^{*\,1-\alpha}}{e_k^*\left(\gamma^*,g_k^*\right)}\right]}-\rho-\delta\right\}, \tag{C.5}$$

where $o\left(\gamma^*,g_k^*\right) := \omega^{1/\mu}\phi^k\left(\frac{1+\mu}{\mu}\right)\left(\rho+\left(\frac{1-S}{S}\right)\gamma^*\right)/\left[Q^{*\,1/\mu}\left(1+\phi^k Q^*\right)\right]$, with, trivially: $\lim_{\omega\to 0} o\left(\gamma^*,g_k^*\right) = \lim_{\omega\to 0} o_\gamma\left(\gamma^*,g_k^*\right) = \lim_{\omega\to 0} o_g\left(\gamma^*,g_k^*\right) = 0$. Finally, the steady-state solution can be computed by two relations between γ^* and g_k^*, namely $\gamma^* = \mathcal{G}\left(g_k^*\right)$ and $g_k^* = \mathcal{F}\left(\gamma^*\right)$, that are directly found by inverting (C.2) and (C.5), using

$$e_k^* = \phi^c\left(A g_k^{*\,1-\alpha}-g_k^*\right)-\left(\phi^c-\phi^k\right)\left(\gamma^*+\delta\right) =: e_k^*\left(\gamma^*,g_k^*\right). \tag{C.6}$$

In particular, the function $g_k^* = \mathcal{F}\left(\gamma^*\right)$ defined in (25) is derived from the following implicit relation:

$$g_k^* = \left\{\frac{\frac{\gamma^*}{S}+\delta+\rho}{\alpha A\left(1-\tau\right)\left[1-o\left(\gamma^*,g_k^*\right)\right]-\left(\frac{1+\mu}{\mu}\right)\left(\frac{\gamma^*}{S}+\delta+\rho\right)\frac{\eta\theta A\phi^k}{e_k^*\left(\gamma^*,g_k^*\right)}}\right\}^{\frac{1}{1-\alpha}}. \tag{C.7}$$

Proof of Proposition 2.
From (C.7), for small values of θ and ω, it is clear that $\mathcal{F}\in C^\infty(\mathbb{R}^*)$, and that $\mathcal{F}$ is an increasing strictly convex function, since $\mathcal{F}(\gamma^*)\to\{[\delta+\rho+\gamma^*/S]/\alpha A(1-\tau)\}^{1/(1-\alpha)}$ when $(\theta,\omega)\to(0,0)$. In addition, $\mathcal{F}\left(0\right) = \left[\frac{\rho+\delta}{\alpha A(1-\tau)}\right]^{1/(1-\alpha)} =: g_k^S > 0$, and $\lim_{\gamma^*\to+\infty}\mathcal{F}\left(\gamma^*\right) = +\infty$. From (23), we notice that $\mathcal{G}\in C^2(]-\infty,\overline{\overline{g}}_k^*[)$, where $\overline{\overline{g}}_k^* := [(\theta+\tau)A-(1-\eta)\theta A/S]^{1/\alpha} > g_k^S = \mathcal{F}(0)\geq 0$,[16] hence, $\overline{\overline{g}}_k^* < (>)\, g_k^B$ if $\eta < (>)\, 1-S$. Besides, $\mathcal{G}$ is a strictly increasing function on $]-\infty,\overline{\overline{g}}_k^*[$, since $\mathcal{G}'(g_k^*) = \frac{\rho(1-\eta)\theta A\alpha S}{g_k^{1-\alpha}\{S[(\theta+\tau)A-g_k^{*\alpha}]-(1-\eta)\theta A\}^2} > 0$. At last, $\lim_{g_k^*\to-\infty}\mathcal{G}\left(g_k^*\right) = 0$, and $\lim_{g_k^*\to\overline{\overline{g}}_k^*}\mathcal{G}\left(g_k^*\right) = +\infty$.

Finally, as $\overline{\overline{g}}_k^* > g_k^S$, using Bolzano's theorem, there are two and only two values of γ^*, denoted by γ_1^* and γ_2^*, such as: $\gamma_i^* > 0$, and $(\mathcal{G}\circ\mathcal{F})(\gamma_i^*) = \gamma_i^*$, $i=1,2$. As BGPs are obtained at the intersection of (23) and (24), we define by $\gamma^{*h} := \max(\gamma_1^*,\gamma_2^*)$ the high BGP solution, and by $\gamma^{*l} := \min(\gamma_1^*,\gamma_2^*)$ the low BGP solution (see Figure 1). □

[16] The case $\overline{\overline{g}}_k^* \leq g_k^S$ is not relevant because there is no equilibrium in this situation.

Appendix D. Effect of deficit and monetization in the long-run.

By introducing (C.2) and (C.6) in (C.5) and defining $v = \phi_k/\phi_c$, we obtain the following implicit relation for γ^*

$$\mathcal{H}(\gamma^*, \theta, \eta) = [1 - o(\gamma^*, \theta, \eta\,)]\,\mathcal{P}(\gamma^*, \theta, \eta) - \rho - \delta - \frac{\gamma^*}{S} = 0, \tag{D.1}$$

where

$$\mathcal{P}(\gamma^*, \theta, \eta) := \frac{(1-\tau)\,\alpha A^{\frac{1}{\alpha}} x(\gamma^*)\,[\sigma(\gamma^*)]^{\frac{1-\alpha}{\alpha}}}{\eta\theta\left(\frac{1+\mu}{\mu}\right) v + x(\gamma^*)}, \tag{D.2}$$

$\sigma(\gamma^*) := \tau - \theta\,[(1-\eta)\,\varepsilon(\gamma^*) - 1]$ and $x(\gamma^*) := 1 - \sigma(\gamma^*) - (1-v)(\gamma^* + \delta)\,A^{\frac{-1}{\alpha}}\,[\sigma(\gamma^*)]^{\frac{\alpha-1}{\alpha}}$.

First, we prove that, for small values of the deficit ratio and transactions costs, $\partial\mathcal{H}(\cdot)/\partial\gamma^* < 0$ in the neighborhood of the high BGP and $\partial\mathcal{H}(\cdot)/\partial\gamma^* > 0$ in the neighborhood of the low BGP. By (D.1), we obtain, if $\omega \to 0$

$$\frac{\partial\mathcal{H}(\cdot)}{\partial\gamma^*} = \frac{\partial o(\cdot)}{\partial\gamma^*}\mathcal{P}(\gamma^*, \theta, \eta) + [1 - o(\cdot)]\,\frac{\partial\mathcal{P}(\cdot)}{\partial\gamma^*} - \frac{1}{S} \to \frac{\partial\mathcal{P}(\cdot)}{\partial\gamma^*} - \frac{1}{S}, \text{ and,} \tag{D.3}$$

$$\frac{\partial\mathcal{H}(\cdot)}{\partial j} = \frac{\partial o(\cdot)}{\partial j}\mathcal{P}(\gamma^*, \theta, \eta) + [1 - o(\cdot)]\,\frac{\partial\mathcal{P}(.)}{\partial j} \to \frac{\partial\mathcal{P}(\cdot)}{\partial j},\ j = \{\theta, \eta\}. \tag{D.4}$$

From (D.2) with $\theta \to 0$, we find, in the neighborhood of the high BGP: $\partial\mathcal{P}(\cdot)/\partial\gamma^*|_{\gamma^*=\gamma^h} = 0$. Thus: $\lim_{(\omega,\theta)\to(0,0)}\{\,\partial\mathcal{H}(\cdot)/\partial\gamma^*|_{\gamma^*=\gamma^h}\} = -1/S < 0$. In addition, since $\mathcal{H} \in C^0([\gamma^l, \gamma^h])$, and (according to Appendix C) $\mathcal{H}(\gamma^*) = 0$ for only two values γ^{*l} and γ^{*h}, $\partial\mathcal{H}(\cdot)/\partial\gamma^*{}_{\gamma^*=\gamma^l} > 0$ in the neighborhood of the low BGP.

Besides, for any $i = \{S, B\}$, we have

$$\left.\frac{\partial\mathcal{P}(\cdot)}{\partial\theta}\right|_{\theta\to 0} = \frac{(1-\tau)\,(\tau A)^{\frac{1}{\alpha}}}{\tau^2}\left[(1-\alpha)\,x(\gamma^i)\,\varepsilon(\gamma^i) - \frac{\alpha\tau v}{x(\gamma^i)}\left(\frac{1+\mu}{\mu}\right)\right](\eta - \bar{\eta}^i), \tag{D.5}$$

$$\frac{\partial P(\cdot)}{\partial\eta} = \frac{(1-\tau)\,\alpha\theta A^{\frac{1}{\alpha}} x(\gamma^*)\,[\sigma(\gamma^*)]^{\frac{1-2\alpha}{\alpha}}}{\left[\eta\theta\left(\frac{1+\mu}{\mu}\right) v + x(\gamma^*)\right]^2} X, \tag{D.6}$$

where $X := \left\{\left[\frac{x'(\gamma^*)}{x(\gamma^*)}\sigma(\gamma^*) + \frac{1-\alpha}{\alpha}\theta\varepsilon(\gamma^*)\right]\eta - \sigma(\gamma^*)\right\}\left(\frac{1+\mu}{\mu}\right) v + \frac{1-\alpha}{\alpha}\varepsilon(\gamma^*)\,x(\gamma^*)$. For $\theta \to 0$, $X \to \frac{1}{\mu}\left[\varepsilon(\gamma^*)\left(\frac{1-\alpha}{\alpha}\right) x(\gamma^*) - \tau v\right](\mu - \bar{\mu})$, so that

$$\text{Sign}\left\{\frac{\partial\mathcal{P}(\cdot)}{\partial\eta}\right\} = \text{Sign}\{X\} = \text{Sign}\{\mu - \bar{\mu}\}. \tag{D.7}$$

Appendix E. Local stability for $\phi^k = 0$.

The Jacobian matrix of system (40) is

$$\mathbf{J}_2^i = \begin{bmatrix} -\xi & 0 & 0 & 0 \\ B_d^i & B_b^i & 0 & b_k^{*i} \\ Q_d^i & Q_b^i & Q_q^i & Q_c^i \\ C_d^i & C_b^i & C_q^i & C_c^i \end{bmatrix}. \tag{E.1}$$

We first define the following derivatives of g_k and r in (38) and (39), namely

$$g_d^i := \left.\frac{\partial g_k}{\partial d_y}\right|_{*i} = \frac{Ay_k^{*i}}{1-\left[\theta+\tau-\alpha(1-\tau)b_k^{*i}\right](1-\alpha)A\left(g_k^{*i}\right)^{-\alpha}},\ g_b^i := \left.\frac{\partial g_k}{\partial b_k}\right|_{*i} = -g_d^i \frac{r^{*i}}{Ay_k^{*i}},\ \text{and,}$$

$$r_g^i := \left.\frac{\partial r}{\partial g_k}\right|_{*i} = \alpha\left(1-\tau\right)\left(1-\alpha\right)A\left(g_k^{*i}\right)^{-\alpha}.$$

It follows that

$$B_d^i := \left.\frac{\partial \dot{b}_k}{\partial d_y}\right|_{*i} = (1-\eta)\left[y_k^{*i} + \theta\left(1-\alpha\right)A\left(g_k^{*i}\right)^{-\alpha} g_d^i\right] - \zeta\left(g_k^{*i}\right) g_d^i b_k^{*i},$$

$$B_b^i := \left.\frac{\partial \dot{b}_k}{\partial b_k}\right|_{*i} = -\gamma^{*i} + (1-\eta)\,\theta\left(1-\alpha\right)A\left(g_k^{*i}\right)^{-\alpha} g_b^i - \zeta\left(g_k^{*i}\right) g_b^i b_k^{*i},$$

$$Q_d^i := \left.\frac{\partial \dot{Q}}{\partial d_y}\right|_{*i} = -\left[\frac{\mu Q^{*i}\left(1+\phi^c Q^{*i}\right)}{1+(1+\mu S)\phi^c Q^{*i}}\right]\left\{\frac{\eta\left[y_k^{*i}+\theta(1-\alpha)A\left(g_k^{*i}\right)^{-\alpha} g_d^i\right]}{\phi^c c_k^{*i}}\left(\frac{Q^{*i}}{\omega}\right)^{\frac{1}{\mu}} + (1-S)\, r_g^i g_d^i\right\},$$

$$Q_b^i := \left.\frac{\partial \dot{Q}}{\partial b_k}\right|_{*i} = -\left[\frac{\mu Q^{*i}\left(1+\phi^c Q^{*i}\right)}{1+(1+\mu S)\phi^c Q^{*i}}\right]\left[\frac{\eta\theta(1-\alpha)A\left(g_k^{*i}\right)^{-\alpha} g_b^i}{\phi^c c_k^{*i}}\left(\frac{Q^{*i}}{\omega}\right)^{\frac{1}{\mu}} + (1-S)\, r_g^i g_b^i\right],$$

$$Q_q^i := \left.\frac{\partial \dot{Q}}{\partial Q}\right|_{*i} = \left[\frac{\mu Q^{*i}\left(1+\phi^c Q^{*i}\right)}{1+(1+\mu S)\phi^c Q^{*i}}\right]\left\{\left(\frac{Q^{*i}}{\omega}\right)^{\frac{1}{\mu}} - \frac{\eta\theta y_k^{*i}}{\omega\mu\phi^c c_k^{*i}}\left(\frac{Q^{*i}}{\omega}\right)^{\frac{1-\mu}{\mu}}\right\},$$

$$Q_c^i := \left.\frac{\partial \dot{Q}}{\partial c_k}\right|_{*i} = \left[\frac{\mu Q^{*i}\left(1+\phi^c Q^{*i}\right)}{1+(1+\mu S)\phi^c Q^{*i}}\right]\left(\frac{\eta\theta y_k^{*i}}{\phi^c (c_k^{*i})^2}\right)\left(\frac{Q^{*i}}{\omega}\right)^{\frac{1}{\mu}},$$

$$C_d^i := \left.\frac{\partial \dot{c}}{\partial d_y}\right|_{*i} = \left(r_g^i g_d^i - \frac{\phi^c Q_d^i}{1+\phi^c Q^{*i}}\right) S c_k^{*i} - \zeta\left(g_k^{*i}\right) g_d^i c_k^{*i},$$

$$C_b^i := \left.\frac{\partial \dot{c}}{\partial b_k}\right|_{*i} = \left(r_g^i g_b^i - \frac{\phi^c Q_b^i}{1+\phi^c Q^{*i}}\right) S c_k^{*i} - \zeta\left(g_k^{*i}\right) g_b^i c_k^{*i},$$

$$C_q^i := \left.\frac{\partial \dot{c}_k}{\partial Q}\right|_{*i} = -\left(\frac{\phi^c Q_q^i}{1+\phi^c Q^{*i}}\right) S c_k^{*i},\ \text{and}\ C_c^i := \left.\frac{\partial \dot{c}}{\partial c_k}\right|_{*i} = -\left(\frac{\phi^c Q_c^i}{1+\phi^c Q^{*i}}\right) S c_k^{*i} + c_k^{*i}.$$

On the Barro BGP, we obtain for $\theta \to 0$

$$\mathbf{J}_1^B = \begin{bmatrix} -\xi & 0 & 0 & 0 \\ (1-\eta)\, y_k^B & -\gamma^B & 0 & 0 \\ Q_d^B & Q_b^B & Q_q^B & 0 \\ C_d^B & C_b^B & C_q^B & c_k^B \end{bmatrix}.$$

Thus the "Barro" BGP is saddle-path stable, with the associated eigenvalues equal to: $-\xi < 0$, $-\gamma^B < 0$, $Q_q^B = \left(\frac{Q^B}{\omega}\right)^{\frac{1}{\mu}}\left[\frac{\mu Q^B\left(1+\phi^c Q^B\right)}{1+(1+\mu S)\phi^c Q^B}\right] > 0$ and $c_k^B > 0$.

Regarding the Solow BGP, the Jacobian matrix is, for "small" deficit values ($\theta \to 0$)

$$\mathbf{J}_1^S = \begin{bmatrix} -\xi & 0 & 0 & 0 \\ B_d^S & -\zeta\left(g_k^S\right) g_b^S b_k^S & 0 & b_k^S \\ Q_d^S & Q_b^S & Q_q^S & 0 \\ C_d^S & C_b^S & C_q^S & c_k^S \end{bmatrix},$$

Hence,

$$\det\left(\mathbf{J}_1^S - \lambda\mathbf{I}\right) = \begin{bmatrix} -\xi-\lambda & 0 & 0 & 0 \\ B_d^S & -\zeta\left(g_k^S\right) g_b^S b_k^S - \lambda & 0 & b_k^S \\ Q_d^S & Q_b^S & Q_q^S - \lambda & 0 \\ C_d^S & C_b^S & C_q^S & c_k^S - \lambda \end{bmatrix}.$$

As the discount rate ρ is small (formally: $\rho \to 0^+$), we have $Q_b^S \to 0$ and $C_b^S \to 0$ and it is true that: $\det\left(\mathbf{J}_1^S - \lambda \mathbf{I}\right) \approx (\xi + \lambda)\left(\zeta\left(g_k^S\right) g_b^S b_k^S + \lambda\right)\left(Q_b^S - \lambda\right)\left(c_k^S - \lambda\right)$. Since $\zeta\left(g_k^S\right) > 0$ and $g_b^S = \frac{-\rho g_k^S}{\alpha g_k^S + \delta(1-\alpha) b_k^S} < 0$,[17] one eigenvalue is negative ($-\xi < 0$) and the three others are positive: $-\zeta\left(g_k^S\right) g_b^S b_k^S > 0$, $Q_q^S = \left(\frac{Q^S}{\omega}\right)^{\frac{1}{\mu}} \left[\frac{\mu Q^S\left(1+\phi^c Q^S\right)}{1+(1+\mu S)\phi^c Q^S}\right] > 0$ and $c_k^S > 0$. It follows that the Solow BGP is unstable.

Appendix F. Local stability for $\phi^k > 0$.

The Jacobian matrix of system (41) is

$$\mathbf{J}_2^i = \begin{bmatrix} -\xi & 0 & 0 & 0 & 0 \\ (1-\eta)\, y_k^{*i} & -\gamma^{*i} & 0 & b_k^{*i} & B_g^i \\ Q_d^i & Q_b^i & r^{*i} + \delta & 0 & Q_g^i \\ C_d^i & C_b^i & C_q^i & c_k^{*i} & C_g^i \\ G_d^i & G_b^i & G_q^i & G_c^i & G_g^i \end{bmatrix}. \tag{F.1}$$

We first define the following derivatives of r in (35) as

$$r_g^i := \left.\frac{\partial r}{\partial g_k}\right|_{*i} = \left[(\theta+\tau)(1-\alpha) A \left(g_k^{*i}\right)^{-\alpha} - 1\right] / b_k^{*i},$$
$$r_b^i := \left.\frac{\partial r}{\partial b_k}\right|_{*i} = \left[g_k^{*i} - (\theta+\tau)\, y_k^{*i}\right] / \left(b_k^{*i}\right)^2.$$

It follows that

$$B_g^i := \left.\frac{\partial \dot{b}_k}{\partial g_k}\right|_{*i} = (1-\eta)\,\theta\left(1+\zeta\left(g_k^{*i}\right)\right) - \zeta\left(g_k^{*i}\right) b_k^{*i},$$
$$Q_d^i := \left.\frac{\partial \dot{Q}}{\partial d_y}\right|_{*i} = \frac{y_k^{*i}}{b_k^{*i}}\left(\frac{1+\phi^k Q^{*i}}{\phi^k}\right);\ Q_b^i := \left.\frac{\partial \dot{Q}}{\partial b_k}\right|_{*i} = r_b^i\left(\frac{1+\phi^k Q^{*i}}{\phi^k}\right),$$
$$Q_g^i := \left.\frac{\partial \dot{Q}}{\partial g_k}\right|_{*i} = \left[r_g^i\left(1+\phi^k Q^{*i}\right) - \alpha(1-\tau)\left(1+\zeta\left(g_k^{*i}\right)\right)\right] / \phi^k,$$
$$C_d^i := \left.\frac{\partial \dot{c}}{\partial d_y}\right|_{*i}\left(\frac{y_k^{*i}}{b_k^{*i}} - \frac{\phi^c Q_d^i}{1+\phi^c Q^{*i}}\right) S c_k^{*i};\ C_b^i := \left.\frac{\partial \dot{c}}{\partial b_k}\right|_{*i} = \left(r_b^i - \frac{\phi^c Q_b^i}{1+\phi^c Q^{*i}}\right) S c_k^{*i},$$
$$C_q^i := \left.\frac{\partial \dot{c}_k}{\partial Q}\right|_{*i} = -\frac{S c_k^{*i}\left(r^{*i}+\delta\right)\phi^c}{\left(1+\phi^c Q^{*i}\right)};\ C_g^i := \left.\frac{\partial \dot{c}_k}{\partial g_k}\right|_{*i} = \left(r_g^i - \frac{\phi^c Q_g^i}{1+\phi^c Q^{*i}}\right) S c_k^{*i} - \zeta\left(g_k^{*i}\right) c_k^{*i},$$
$$G_d^i := \left.\frac{\partial \dot{g}_k}{\partial d_y}\right|_{*i} = \frac{1}{\phi^k \zeta\left(g_k^{*i}\right)}\left\{\eta A g_k^{*i\,1-\alpha}\left(Q^{*i}/\omega\right)^{1/\mu} + \left(\frac{y_k^{*i}}{b_k^{*i}} + \frac{1}{\mu}\frac{Q_d^i}{Q^{i*}}\right) e_k^{*i} - \left(\phi^c - \phi^k\right) C_d^i\right\},$$
$$G_b^i := \left.\frac{\partial \dot{g}_k}{\partial b_k}\right|_{*i} = \frac{1}{\phi^k \zeta\left(g_k^{*i}\right)}\left\{\left(r_b^i + \frac{1}{\mu}\frac{Q_b^i}{Q^{*i}}\right) e_k^{*i} - \left(\phi^c - \phi^k\right) C_b^i\right\},$$
$$G_q^i := \left.\frac{\partial \dot{g}_k}{\partial Q}\right|_{*i} = \frac{1}{\phi^k \zeta\left(g_k^{*i}\right)}\left\{\frac{\eta\theta A g_k^{*i\,1-\alpha}}{\mu\omega}\left(\frac{Q^{*i}}{\omega}\right)^{\frac{1-\mu}{\mu}} + \left[\frac{1}{\mu}\frac{Q_q^i}{Q^{i*}} - \left(\frac{Q^{*i}}{\omega}\right)^{\frac{1}{\mu}}\right] e_k^{*i} - \left(\phi^c - \phi^k\right) C_q^i\right\},$$
$$G_c^i := \left.\frac{\partial \dot{g}_k}{\partial c_k}\right|_{*i} = \frac{1}{\phi^k \zeta\left(g_k^{*i}\right)}\left\{\left(r^{*i} - R^{*i} - \gamma^{*i} - c_k^{*i}\right)\left(\phi^c - \phi^k\right) + e_k^{*i}\right\},$$
$$G_g^i := \left.\frac{\partial \dot{g}_k}{\partial g_k}\right|_{*i} = \frac{1}{\phi^k \zeta\left(g_k^{*i}\right)}\left\{\eta\theta\left(1+\zeta\left(g_k^{*i}\right)\right)\left(\frac{Q^{*i}}{\omega}\right)^{\frac{1}{\mu}} + \left(r_g^i - \zeta\left(g_k^{*i}\right) + \frac{Q_g^i}{\mu Q^{*i}}\right) e_k^{*i} \right.$$
$$\left. + \left(r^{*i} - \gamma^{*i} - R^{*i}\right)\phi^k \zeta\left(g_k^{*i}\right) - \left(\phi^c - \phi^k\right) C_g^i\right\}.$$

[17] Notice that: $1 - \left[\tau - \alpha(1-\tau) b_k^S\right](1-\alpha) A \left(g_k^S\right)^{-\alpha} = \left\{(1-\alpha)\left[g_k^S - \tau A (g_k^S)^{1-\alpha} + (\rho+\delta) b_k^S\right] + \alpha g_k^S\right\} / g_k^S$.

To establish formal results, we study the local dynamics of the two BGPs for small deficit values ($\theta \to 0$), namely in the neighborhood of the "Barro" BGP and of the "Solow" BGP, respectively. Besides, to simplify calculations, we consider that $\phi^k = \phi^c = \phi > 0$, i.e. $e_k^{*i} = \phi\left(y_k^{*i} - g_k^{*i}\right) = \phi\left(c_k^{*i} + \gamma^{*i} + \delta\right)$. We first define the following matrix

$$\mathbf{H}_2^i = \begin{bmatrix} r^i + \delta & 0 & Q_g^i \\ C_q^i & c_k^i & C_g^i \\ G_q^i & G_c^i & G_g^i \end{bmatrix},$$

where: $\det\left(\mathbf{H}_2^i\right) = \left(r^i + \delta\right)\left(c_k^i G_g^i - G_c^i C_g^i\right) + \left(C_q^i G_c^i - c_k^i G_q^i\right) Q_g^i$, namely, after straightforward arithmetics

$$\det\left(\mathbf{H}_2^i\right) = \frac{c_k^{*i} e_k^{*i}}{\zeta\left(g_k^{*i}\right)}\left[\left(r^{*i} + \delta\right)(1 - S)\, r_g^i + Q_g^i\left(\frac{Q^{*i}}{\omega}\right)^{1/\mu}\right],\ i = \{S, B\}.$$

For $\theta \to 0$, $r_g^i < 0$, hence $Q_g^i < 0$, and it follows that $\text{Sign}\{\det\left(\mathbf{H}_2^i\right)\} = -\text{Sign}\{\zeta(g_k^{*i})\}$ under the condition $S < \bar{S} := 1 + [Q_g^i/(r^{*i} + \delta) r_g^i](Q^{*i}/\omega)^{1/\mu} > 1$ that we suppose to be verified. Moreover, the characteristic equation of $\mathbf{H}_2^i$ writes

$$P(\lambda) = \left(r^{*i} + \delta - \lambda\right)\left[\left(c_k^{*i} - \lambda\right)\left(G_g^i - \lambda\right) - C_g^i G_c^i\right] + C_q^i Q_g^i G_c^i - G_q^i Q_g^i\left(c_k^{*i} - \lambda\right),$$

namely $P(\lambda) = -\lambda^3 + p_1^i \lambda^2 - p_2^i \lambda + \det\left(\mathbf{H}_2^i\right) = 0$, where $p_1^i := r^{*i} + \delta + c_k^{*i} + G_g^i$, $p_2^i := \left(r^{*i} + \delta\right)\left(c_k^{*i} + G_g^i\right) + \left(c_k^{*i} G_g^i - C_g^i G_c^i\right) - G_q^i Q_g^i$, and we can notice that

$$\begin{aligned}\zeta\left(g_k^{*i}\right) p_2^i :=& \left(r_g^i - \zeta\left(g_k^{*i}\right)\right)\left(r^{*i} + \delta\right)\left(\gamma^{*i} + \delta\right) + r_g^i\left(r^{*i} + \delta\right) c_k^{*i} \\ &+ \left[\left[(r_g^i + \frac{Q_g^i}{\mu Q^{*i}}) - \frac{\alpha(1 - \tau)\left(1 + \zeta\left(g_k^{*i}\right)\right)}{1 + \phi Q^{*i}} S\right] c_k^{*i} + Q_g^i\left(\frac{Q^{*i}}{\omega}\right)^{\frac{1}{\mu}}\right] e_k^{*i},\end{aligned}$$

so that $\text{Sign}\left\{p_2^i\right\} = -\text{Sign}\left\{\zeta\left(g_k^{*i}\right)\right\}$.[18]

In addition, we define: $P'(\lambda) = -3\lambda^2 + 2p_1^i \lambda - p_2^i$, with $P'(0) = -p_2^i$ and $P'(\bar{\lambda}_1) = P'(\bar{\lambda}_2) = 0$ where $\bar{\lambda}_{1,2} = [p_1^i \pm \sqrt{(p_1^i)^2 - 3p_2^i}]/3$.

In the neighborhood of the high BGP, for small deficit values ($\theta \to 0$), the Jacobian matrix (E.1) rewrites

$$\mathbf{J}_2^B = \begin{bmatrix} -\xi & 0 & 0 & 0 & 0 \\ (1 - \eta)\, y_k^B & -\gamma^B & 0 & 0 & 0 \\ Q_d^B & Q_b^B & r^B + \delta & 0 & Q_g^B \\ C_d^B & C_b^B & C_q^B & c_k^B & C_g^B \\ G_d^B & G_b^B & G_q^B & G_c^B & G_g^B \end{bmatrix}. \tag{F.2}$$

Thus, $\mathbf{J}_2^B$ has two negative eigenvalues, namely $\lambda_1 = -\xi$ and $\lambda_2 = -\gamma^B$. Furthermore: $\det\left(\mathbf{J}_2^B\right) = \xi\, \gamma^B \det\left(\mathbf{H}_2^B\right)$, with $\text{Sign}\left\{\det\left(\mathbf{H}_2^B\right)\right\} = -\text{Sign}\left\{\zeta\left(g_k^B\right)\right\}$. Consequently, $\mathbf{J}_2^B$ has at least 3 negative eigenvalues, if $\zeta\left(g_k^B\right) > 0$, and the high BGP is undetermined.

[18] Remark that the term into brackets is positive. Effectively, $r_g^i < 0$, $Q_g^i < 0$, and , if $\zeta\left(g_k^h\right) < 0$, we have $r_g^h \to -\alpha/b_k^h << \zeta\left(g_k^h\right)$ since b_k^h is very small.

If $\zeta(g_k^B) < 0$, observe that: $p_1^B := r^B - \gamma^B + (r_g^B + Q_g^B/\mu Q^{*B})(c_k^B + \gamma^B + \delta)/\zeta(g_k^B) > 0$ and $p_2^B > 0$. Thus: $\bar{\lambda}_j = [p_1^i \pm \sqrt{(p_1^i)^2 - 3p_2^i}]/3 > 0$, $j = \{1,2\}$. Since $P(0) = \det(\mathbf{H}_2^B) > 0$ and $P'(0) = -p_2^B < 0$, $\mathbf{H}_2^B$ contains three positive eigenvalues (case A of Figure A.1) and the high BGP is well determined. By continuity, these properties are verified for positive (but low) deficit values. This proves point (ii) of Proposition 6.

In the neighborhood of the low BGP, for small deficit values ($\theta \to 0$), the Jacobian matrix (E.1) becomes

$$\mathbf{J}_2^S = \begin{bmatrix} -\xi & 0 & 0 & 0 & 0 \\ (1-\eta)\,y_k^S & 0 & 0 & b_k^S & -\zeta\left(g_k^S\right)b_k^S \\ Q_d^S & -\rho^2\tilde{Q}_b^S & \rho+\delta & 0 & Q_g^S \\ 0 & 0 & C_q^S & c_k^S & C_g^S \\ G_d^S & -\rho^2\tilde{G}_b^S & G_q^S & G_c^S & G_g^S \end{bmatrix}. \tag{F.3}$$

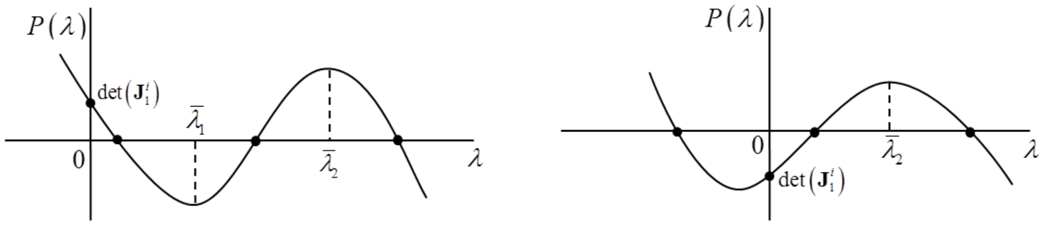
Hence;

$$\mathbf{J}_2^S - \lambda\mathbf{I} = \begin{bmatrix} -\xi-\lambda & 0 & 0 & 0 & 0 \\ (1-\eta)\,y_k^S & -\lambda & 0 & b_k^S & -\zeta\left(g_k^S\right)b_k^S \\ Q_d^S & -\rho^2\tilde{Q}_b^S & \rho+\delta-\lambda & 0 & Q_g^S \\ 0 & 0 & C_q^S & c_k^S-\lambda & C_g^S \\ G_d^S & -\rho^2\tilde{G}_b^S & G_q^S & G_c^S & G_g^S-\lambda \end{bmatrix},$$

thus

$$\det\left(\mathbf{J}_2^S - \lambda\mathbf{I}\right) = \det\begin{bmatrix} -\xi-\lambda & 0 & 0 & 0 & 0 \\ (1-\eta)\,y_k^S - Q_d^S & \rho^2\tilde{Q}_b^S-\lambda & \lambda-\rho-\delta & b_k^S & -Q_g^S-\zeta\left(g_k^S\right)b_k^S \\ Q_d^S & -\rho^2\tilde{Q}_b^S & \rho+\delta-\lambda & 0 & Q_g^S \\ 0 & 0 & C_q^S & c_k^S-\lambda & C_g^S \\ G_d^S & -\rho^2\tilde{G}_b^S & G_q^S & G_c^S & G_g^S-\lambda \end{bmatrix}.$$

As the discount rate ρ is small (formally: $\rho^2 \to 0^+$), it is true that: $\det(\mathbf{J}_2^S - \lambda\mathbf{I}) \approx -(\xi+\lambda)(\rho^2\tilde{Q}_b^S - \lambda)\det(\mathbf{H}_2^S - \lambda\mathbf{I})$. Therefore, one eigenvalue is negative ($\lambda_1 = -\mu < 0$) and another one is positive ($\lambda_2 \approx \rho^2\tilde{Q}_b^S \to 0^+$). Furthermore, since $\zeta(g_k^S) > 0$, $P(0) = \det(\mathbf{H}_2^S) < 0$ and $\mathbf{H}_2^S$ contains either one or three negative eigenvalues. However, since $P'(0) = -p_2^S > 0$, $P'(\lambda) = 0$ for one positive value (say, $\bar{\lambda}_2 = [p_1^i + \sqrt{(p_1^i)^2 - 3p_2^i}]/3 > 0$), thus there is necessarily at least one positive eigenvalue (see the case B of Figure A.1). It follows that the low BGP is well determined, with $\mathbf{J}_2^S$ containing exactly two negative and three positive eigenvalues, which proves point (i) of Proposition 6.



Case A **Case B**

Figure A.1: Sign of eigenvalues